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Stochastic Differential Equations – Some New Ideas

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Abstract

In this paper we present a general method to study stochastic equations for a broader class of driving noises. We explain the main principles of this approach in the case of stochastic differential equations driven by a Wiener process. As a result we construct strong solutions of Itô equations with discontinuous and even functional coefficients. We point out that our construction of solutions does not rely on a pathwise uniqueness argument. Further we find that solutions of a larger class of Itô diffusions actually live in a Fréchet space, which is substantially smaller than the Meyer-Watanabe test function space.

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1 Introduction

We develop a general approach to study stochastic equations for a broader class of driving noises. The range of applications pertains e.g. to strong solutions of stochastic differential equations (SDE's) driven by additive processes, fractional Lévy processes, infinite dimensional SDE's, stochastic partial differential equations in the anticipating sense or not. See Section 5, where we discuss the applicability of our method. In this paper we analyze strong solutions of Itô equations with discontinuous coefficients. We also permit the coefficients to be functional, that is we allow the coefficients in the Itô equations to depend on the past of the solutions. Further we provide a tool to construct strong solutions, that is solutions which are functionals of the driving noise, in an explicit way. More precisely we start out with a generalized stochastic process, which is explicitly defined in a stochastic distribution space. Then, using an approximation argument we directly verify this process as a solution of the corresponding stochastic equation. We emphasize that our technique does not resort to a pathwise uniqueness argument. This technique also leads to estimates, which can be useful for the numerical analysis of solutions. We think that our approach may contribute to a better understanding of stochastic equations.

In this paper we want to illustrate the main principles of our method on the basis of SDE's driven by a Wiener process. See also [P1], [P2] for other applications.

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Consider the functional SDE

$$dX_t = b(t, X.)dt + \sigma(t, X.)dB_t, 0 \le t \le T,\ X_0 = \ x \in \mathbb{R}^d, \tag{1}$$

where B_t is a $d-$dimensional Brownian motion with respect to a filtration $\mathcal{F}_t$, generated by B_t on a probability space $(\Omega, \mathcal{F}, \pi)$. Further the drift coefficient $b : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ and dispersion coefficient $\sigma : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^{d\times d}$ are assumed to be progressively measurable functionals. Here $\mathcal{W} = C([0,T])$ is the Wiener space.

It is well known that if

$$\|b(t,\phi)\| + \|\sigma(t,\phi)\| \le C(1 + \max_{0\le s\le t} \|\phi(s)\|),\ \phi \in \mathcal{W}^d,\ 0 \le t \le T \tag{2}$$

$$\|b(t,\phi) - b(t,\psi)\| + \|\sigma(t,\phi) - \sigma(t,\psi)\| \le D(\max_{0\le s\le t} \|\phi(s) - \psi(s)\|),\ \phi, \psi \in \mathcal{W}^d,\ 0 \le t \le T, \tag{3}$$

then there exists a unique strong solution X_t of (1), that is a continuous $\mathcal{F}_t-$adapted process X_t solving (1). Moreover we have that

$$E\left[\int_0^T X_t^2 dt\right] < \infty.$$

Regarding the existence of strong solutions of SDE's two questions naturally arise:

(A) Are there still (unique) strong solutions of SDE's, when the coefficients of (1) are chosen to be irregular, that is e.g. non-Lipschitzian, discontinuous or not Sobolev differentiable?

(B) Can we say more about the smoothness of solutions, even in the case of irregular coefficients ? For example are the solutions contained in a "small" subspace of $L^p(\pi)$?

Surprisingly, a scarce number of authors in literature deals with these important problems:

Let us have a closer look at problem (A).

(A): Given the deterministic ordinary differential equation

$$\frac{dX_t}{dt} = b(t, X_t),\ X_0 = x$$

remember that a solution may not be unique or even not exist, if b is non-Lipschitzian.

However, adding a white noise term to the right hand side of this equation, that is

$$X_t = x + \int_0^t b(s, X_s)ds + \varepsilon B_t$$

elicits the amazing fact that a unique global strong strong solution exists for all $\varepsilon > 0$, when b is e.g. bounded and measurable- regardless how small ε is. This result of Zvonkin [Zv] can be considered a milestone of the theory of SDE's. See also Veretennikov [V], where the multidimensional case is treated. The authors use estimates of solutions of parabolic partial differential equations to construct weak solutions and obtain strong ones by means of pathwise uniqueness.

Recently- in a more general setting- the latter results have been improved by Gyöngy, Martínez [GM] and Krylov, Röckner [KR], who presume L^p-integrability on the drift coefficient to ensure existence and uniqueness of strong solutions. The authors first derive weak solutions. For this purpose Gyöngy and Martínez invoke the Skorohod embedding, whereas Krylov and Röckner resort to an argument of Portenko [Po], which utilizes Girsanov's change of measure. Then they verify pathwise uniqueness to establish strong solutions. Another method resting on an Euler scheme of approximation and strong uniqueness is presented in Gyöngy, Krylov [GK]. See also [FZ].

As mentioned our method to find strong solutions is not based on a pathwise uniqueness (or strong uniqueness) argument and even applies to Itô equations with functional coefficients. We remark that as far as we can see the framework of the above authors cannot be employed to study functional SDE's. The reason is that the authors' techniques involve specific estimates on the Euclidean space. See e.g. [GM, Lemma 3.1], where an estimate of Krylov [K2] for semimartingales is called on. Results on properties of strong solutions of Itô equations with regular (i.e. Lipschitz continuous) functional coeffcients can be found e.g. in [KS] and [H]. Let us mention that the paper of [KR] also focuses on the aspect of equations with singular (and time dependent) drift. However, it is not clear for us to which extent our approach may cover this important issue .

Strong solutions of SDE's with irregular coefficients are important from the viewpoint of applications. For example important applications are stimulated by the following fields:

(i) Statistical mechanics of infinite particle systems: There the stochastic dynamics of particles is determined by a Brownian motion with irregular (singular) drift. In this case it is desirable to look for solutions which are functions of the Brownian motion, that is strong solutions. See Krylov, Röckner [KR].

(ii) Stochastic control theory. See Krylov [K1].

We now turn our attention to the question of smoothness of solutions:

(B) Watanabe [W] showed- also in a more general setting- that if b and σ are time-homogeneous and $b_i, \sigma_{ij} \in C_0^\infty(\mathbb{R}^d)$ (i.e. the space of smooth functions with compact support) the coordinates of the solution X_t of (1) will belong to the Meyer-Watanabe test function space $\mathbb{D}_\infty$, that is

$$X_t^{(i)} \in \mathbb{D}_\infty$$

for all t and $i = 1, ..., d$. Recall that $\mathbb{D}_\infty$ is a dense subspace of $L^2(\pi)$ endowed with a topology given by the seminorms

$$\|F\|_{p,k} = \left(E\left[|F|^p\right] + \sum_{j=1}^{k} E\left[\left\| D_\cdot^j F \right\|_{L^2([0,T]^j)}^p \right] \right)^{\frac{1}{p}}, \ k \in \mathbb{N}, \ p \geq 1, \tag{4}$$

with

$$D_{t_1,...,t_j}^j F(\omega) := D_{t_1} D_{t_2} ... D_{t_j} F(\omega)$$

for $F \in \mathbb{D}_\infty$, where D_t stands for the Malliavin derivative. See e.g. [N] for details. Closely related results to [W] were attained by Stroock [S].

The stochastic concept of smoothness is important in a variety of fields. One encounters this issue e.g. in the following areas:

(i) Mathematical finance: Hedging of contingent claims with stocks, whose price dynamics is modelled by a SDE. There one uses e.g. the Clark-Haussmann-Ocone theorem to determine the closest hedge of a claim with the help of the Malliavin derivative. See e.g. [KO].

(ii) SDE theory: Analysis of the regularity of probability laws of solutions. See [M].

(iii) Monte Carlo methods: Probabilistic method for the numerical computation of risk measures like Greeks in mathematical finance. See [FLLLT].

The objective of the paper is twofold: Firstly we shall lay the foundations for a new approach to strong solutions of stochastic equations. We give the main principles of this general method by analyzing a special case of stochastic equations, namely the SDE (1). Secondly we shall address the above problems (A) and (B). It turns out e.g. that strong solutions of a larger class of Itô equations with irregular coefficients are Malliavin differentiable. Furthermore, we find that strong solutions of a richer class of non-degenerate Itô equations actually live in a Fréchet space $\mathcal{C}_{q,\infty}$ which is substantially smaller than the Meyer-Watanabe test function space $\mathbb{D}_\infty$.

Our approach to stochastic equations involves techniques from Malliavin calculus and white noise analysis.

The paper is organized as follows: In Section 2 we give the framework of our paper. Here we review basic concepts of Gaussian white noise theory, that is we define e.g. the $S-$transform on the Hida space and introduce some spaces of smooth and generalized random variables. In Section 3 we illustrate our method for the SDE (1). In Section 4 we give a construction of solutions in the space $\mathcal{C}_{q,\infty} \subsetneq \mathbb{D}_\infty$. Our main results are Theorem 17, 18, 19 and 27. Section 5 concludes with a discussion of our method.

2 Framework

In this Section we recapitulate some basic concepts of Gaussian white noise analysis. This machinery will be invoked in the next Sections to construct (smooth) solutions of SDE's. For more information about Gaussian white noise analysis we encourage the reader to resort to the excellent books of [HKPS], [O] and [Ku]. See also [HØUZ] for applications to stochastic partial differential equations. As for foundations of a non-Gaussian white noise theory we refer to [KSS]. See also [LøP] in the case of Lévy noise.

In the sequel we aim at working with two different types of stochastic test function and distribution spaces.

2.1 The stochastic test function and distribution space of Hida

We briefly describe the construction of the Hida stochastic test function and distribution space on $\mathbb{R}^d$. Denote by $\mathcal{S}(\mathbb{R})$ the space of rapidly decreasing functions on $\mathbb{R}$ and by $\mathcal{S}'(\mathbb{R})$ its topological dual. Since $\mathcal{S}'(\mathbb{R})$ is a conuclear space the celebrated Bochner-Minlos theorem guarantees the existence of a unique probability measure π on $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ (Borel $\sigma-$algebra of $\mathcal{S}'(\mathbb{R})$), whose characteristic function is given by

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle\omega,\phi\rangle}\pi(d\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}$$

for $\phi \in \mathcal{S}(\mathbb{R})$, where $\langle \omega, \phi\rangle$ is the action of $\omega \in \mathcal{S}'(\mathbb{R})$ on $\phi \in \mathcal{S}(\mathbb{R})$. We define on

$$(\mathcal{S}', \mathcal{B}) := \left(\prod_{i=1}^{d} \mathcal{S}'(\mathbb{R}), \otimes_{i=1}^{d}\mathcal{B}(\mathcal{S}'(\mathbb{R}))\right)$$

the *d-dimensional white noise probability measure* μ as the product measure

$$\mu = \otimes_{i=1}^{d}\pi.$$

For $\omega = (\omega_1, ..., \omega_d) \in \mathcal{S}'$ and $\phi = (\phi^{(1)}, ..., \phi^{(d)}) \in (\mathcal{S}(\mathbb{R}))^d$ define the exponential functional

$$\widetilde{e}(\phi, \omega) = \exp\left(\langle \omega, \phi\rangle - \frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R};\mathbb{R}^d)}\right),$$

where $\langle \omega, \phi\rangle := \sum_{i=1}^{d}\langle \omega_i, \phi_i\rangle$. Denote by $\left((\mathcal{S}(\mathbb{R}))^d\right)^{\widehat{\otimes} n}$ the $n-$th completed symmetric tensor product of $(\mathcal{S}(\mathbb{R}))^d$ with itself. Since $\widetilde{e}(\phi, \omega)$ is holomorphic in ϕ around zero, it can be expanded into a power series. More precisely there exist generalized Hermite polynomials $H_n(\omega) \in \left(\left((\mathcal{S}(\mathbb{R}))^d\right)^{\widehat{\otimes} n}\right)'$ such that

$$\widetilde{e}(\phi, \omega) = \sum_{n\geq 0}\frac{1}{n!}\left\langle H_n(\omega), \phi^{\otimes n}\right\rangle \tag{5}$$

for ϕ in a certain neighbourhood of zero in $(\mathcal{S}(\mathbb{R}))^d$. It can be shown that

$$\left\{\left\langle H_n(\omega), \phi^{(n)}\right\rangle : \phi^{(n)} \in \left((\mathcal{S}(\mathbb{R}))^d\right)^{\widehat{\otimes} n},\ n \in \mathbb{N}_0\right\} \tag{6}$$

forms a total set of $L^2(\mu)$. Furthermore, for all n, m, $\phi^{(n)} \in \left((\mathcal{S}(\mathbb{R}))^d\right)^{\widehat{\otimes} n}$, $\psi^{(m)} \in \left((\mathcal{S}(\mathbb{R}))^d\right)^{\widehat{\otimes} m}$ the orthogonality relation

$$\int_{\mathcal{S}'}\left\langle H_n(\omega), \phi^{(n)}\right\rangle\left\langle H_m(\omega), \psi^{(m)}\right\rangle\mu(d\omega) = \delta_{n,m}n!\left(\phi^{(n)}, \psi^{(n)}\right)_{L^2(\mathbb{R}^n;(\mathbb{R}^d)^{\otimes n})} \tag{7}$$

is valid. Using (7) and a density argument we can extend $\left\langle H_n(\omega), \phi^{(n)}\right\rangle$ to act on $\phi^{(n)} \in L^2(\mathbb{R}^n; (\mathbb{R}^d)^{\otimes n})$ for ω a.e. Note that $\left\langle H_n(\omega), \phi^{(n)}\right\rangle$ can be viewed as a $n-$fold iterated stochastic integral of functions $\phi^{(n)} \in L^2(\mathbb{R}^n; (\mathbb{R}^d)^{\otimes n})$ with respect to a $d-$dimensional Brownian motion

$$B_t = \left(B_t^{(1)}, ..., B_t^{(d)}\right) \tag{8}$$

defined on our white noise space

$$(\Omega, \mathcal{F}, \mu) = \left(\mathcal{S}', \mathcal{B}, \mu\right). \tag{9}$$

Denote by $\widehat{L}^2(\mathbb{R}^n; (\mathbb{R}^d)^{\otimes n})$ the space of square integrable functions $f(x_1, ..., x_n) \in (\mathbb{R}^d)^{\otimes n}$ being symmetric in the variables $x_1, ..., x_n$. Then one infers from (5), (6) and (7) the Wiener-Itô chaos representation property of square integrable Brownian functionals: For all $F \in L^2(\mu)$ there exists a unique sequence of $\phi^{(n)} \in \widehat{L}^2(\mathbb{R}^n; (\mathbb{R}^d)^{\otimes n})$ such that

$$F(\omega) = \sum_{n \geq 0} \left\langle H_n(\omega), \phi^{(n)} \right\rangle \tag{10}$$

for ω a.e. Moreover, we have the Itô-isometry

$$\|F\|_{L^2(\mu)}^2 = \sum_{n \geq 0} n! \left\| \phi^{(n)} \right\|_{L^2(\mathbb{R}^n; (\mathbb{R}^d)^{\otimes n})}^2. \tag{11}$$

We carry on constructing the Hida stochastic test function and distribution space based on the Wiener-Itô chaos expansion (10). To this end let

$$A = 1 + t^2 - \frac{d^2}{dt^2} \tag{12}$$

be the selfadjoint operator with maximal domain $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ and define $A^d = (A, ..., A)$. By invoking a second quantization argument we define the *Hida stochastic test function space* $(\mathcal{S})$ to consist of all $f = \sum_{n \geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle \in L^2(\mu)$ such that

$$\|f\|_{0,p}^2 := \sum_{n \geq 0} n! \left\| \left((A^d)^{\otimes n} \right)^p \phi^{(n)} \right\|_{L^2(\mathbb{R}^n; (\mathbb{R}^d)^{\otimes n})}^2 < \infty \tag{13}$$

for all $p \geq 0$. The space $(\mathcal{S})$ is a nuclear Fréchet algebra equipped with a topology induced by the seminorms $\|\cdot\|_{0,p}$, $p \geq 0$. It can be e.g. seen from (5) that

$$\widetilde{e}(\phi, \omega) \in (\mathcal{S}) \tag{14}$$

for all $\phi \in (\mathcal{S}(\mathbb{R}))^d$.

Further we introduce the *Hida stochastic distribution space* $(\mathcal{S})^*$ as the topological dual of $(\mathcal{S})$. So we obtain the Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*.$$

An important property of the Hida distribution space $(\mathcal{S})^*$ is that it accomodates the *white noise* of the coordinates of the $d-$dimensional Brownian motion B_t. That is the time derivatives

$$W_t^{(i)} := \frac{d}{dt} B_t^{(i)} \in (\mathcal{S})^*, \ i = 1, ..., d \tag{15}$$

in the sense of the topology of $(\mathcal{S})^*$.

The S-transform as a fundamental concept of white noise distribution theory serves as a tool to characterize elements of the Hida test function and distribution space. See [PS]. The $S-$transform of a $\Phi \in (\mathcal{S})^*$, denoted by $S(\Phi)$, is defined as the dual pairing

$$S(\Phi)(\phi) = \langle \Phi, \widetilde{e}(\phi, \omega) \rangle \tag{16}$$

for $\phi \in (\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d$ ($\mathcal{S}_{\mathbb{C}}(\mathbb{R})$ the complexification of $\mathcal{S}(\mathbb{R})$). The $S-$transform is injective, that is, if

$$S(\Phi) = S(\Psi) \text{ for } \Phi, \Psi \in (\mathcal{S})^*$$

then

$$\Phi = \Psi.$$

One verifies e.g. that

$$S(W_t^{(i)})(\phi) = \phi^{(i)}(t), \; i = 1, ..., d \tag{17}$$

for $\phi = (\phi^{(1)}, ..., \phi^{(d)}) \in (\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d$.

Finally we give the important definition of the *Wick* or *Wick-Grassmann product*, which can be considered a tensor algebra multiplication on the Fock space. The Wick product of two distributions $\Phi, \Psi \in (\mathcal{S})^*$, denoted by $\Phi \diamond \Psi$, is the unique element in $(\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi) S(\Psi)(\phi) \tag{18}$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d$. As an example one finds that

$$\left\langle H_n(\omega), \phi^{(n)} \right\rangle \diamond \left\langle H_m(\omega), \psi^{(m)} \right\rangle = \left\langle H_{n+m}(\omega), \phi^{(n)} \widehat{\otimes} \psi^{(m)} \right\rangle \tag{19}$$

for $\phi^{(n)} \in \left((\mathcal{S}(\mathbb{R}))^d \right)^{\widehat{\otimes} n}$, $\psi^{(m)} \in \left((\mathcal{S}(\mathbb{R}))^d \right)^{\widehat{\otimes} m}$. The latter and (5) imply that

$$\widetilde{e}(\phi, \omega) = \exp^{\diamond}(\langle \omega, \phi \rangle) \tag{20}$$

for $\phi \in (\mathcal{S}(\mathbb{R}))^d$. The Wick exponential $\exp^{\diamond}(X)$ of a $X \in (\mathcal{S})^*$ is defined as

$$\exp^{\diamond}(X) = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}, \tag{21}$$

where $X^{\diamond n} = X \diamond ... \diamond X$.

2.2 Spaces of smooth and generalized random variables

As announced in the Introduction we persue the construction of subspaces of $L^2(\mu)$, in which strong solutions of a "richer" class of stochastic differential equations live. In searching for appropriate candidates of such spaces we observe that the Hida test function space $(\mathcal{S})$ is too small to contain solutions of SDE's, since e.g. for $d = 1$ the kernels of their chaos expansion fail to be in $\mathcal{S}(\mathbb{R}^n)$. Another dual pair of spaces, which has proven to be useful for the analysis of strong solutions, is the Meyer-Watanabe test function and distribution space $(\mathbb{D}_{\infty}, \mathbb{D}_{-\infty})$. Although $\mathbb{D}_{\infty}$ comprises solutions of non-degenerate SDE's, it seems to be difficult to establish characterization theorems for $(\mathbb{D}_{\infty}, \mathbb{D}_{-\infty})$. Several attempts in literature have been made to overcome this deficiency: For example the authors [LM], [ÜZ] and [PT] study a dual pair $(\mathcal{G}, \mathcal{G}^*)$. Here the test function space $\mathcal{G}$ is constructed by means of exponential weights of the Ornstein-Uhlenbeck operator. In [PT] a sufficient criterion in terms of the S-transform is provided to characterize $(\mathcal{G}, \mathcal{G}^*)$. Further in [GKS] the authors introduce a scale of spaces of smoothed and generalized random variables including $(\mathcal{G}, \mathcal{G}^*)$ as a special case, where a characterization of $\mathcal{G}$ and $\mathcal{G}^*$ via the Bargmann-Segal space is given. Unfortunately the above authors do not solve the problem, whether $\mathcal{G}$ is rich enough to carry solutions of a broader class of SDE's. In this Section we shall devise a space of smooth random variables $\mathcal{C} = \underset{q \longrightarrow \infty}{proj\ lim} \mathcal{C}_q$ which is closely related to $\mathcal{G}$. In Section 4 we will show that the spaces $\mathcal{C}_q$ actually comprise a larger class of solutions of SDE's. In the sequel we shall focus on the dual pairs $(\mathcal{G}, \mathcal{G}^*)$ and $(\mathcal{C}, \mathcal{C}^*)$. Let us first pass in review the definition and basic properties of $(\mathcal{G}, \mathcal{G}^*)$. See [PT].

Denote by N the number operator or Ornstein-Uhlenbeck operator, which acts on elements of $L^2(\mu)$ by multiplying the $n-$th homogeneous chaos with $n \in \mathbb{N}_0$.

The *space of smooth random variables* $\mathcal{G}$ is defined as the collection of all

$$f = \sum_{n \geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle \in L^2(\mu)$$

such that

$$\|f\|_q^2 := \left\| e^{qN} f \right\|_{L^2(\mu)}^2 < \infty$$

for all $q \geq 0$. The latter condition is equivalent to

$$\|f\|_q^2 = \sum_{n \geq 0} n! e^{2qn} \left\| \phi^{(n)} \right\|_{L^2(\mathbb{R}^n; (\mathbb{R}^d)^{\otimes n})}^2 < \infty \tag{22}$$

for all $q \geq 0$. The space $\mathcal{G}$ is endowed with the topology given by the family of norms $\|\cdot\|_q$, $q \geq 0$.

The *space of generalized random variables* $\mathcal{G}^*$ is the topological dual of $\mathcal{G}$.

It turns out that $\mathcal{G}$ is a nuclear Fréchet algebra with respect to the pointwise multiplication of functions. See [LM], [ÜZ], [PT]. Next consider the norms

$$|||f|||_{p,k} := \left\| (1+N)^{\frac{k}{2}} f \right\|_{L^p(\mu)}, \; k \in \mathbb{N}, \; p \geq 1 \tag{23}$$

on $\mathbb{D}_\infty$. Note that the norms in (23) are equivalent to those in (4). Let $k \in \mathbb{N}$, $p \geq 1$. Then using the hypercontractivity theorem of Nelson (see e.g. [N]) and the spectral theorem entails the following estimate: We can choose $q \geq 0$ large enough such that

$$\begin{aligned} |||f|||_{p,k} &= \left\| e^{-qN} (1+N)^{\frac{k}{2}} e^{qN} f \right\|_{L^p(\mu)} \\ &\leq \left\| e^{-\frac{q}{2}N} (1+N)^{\frac{k}{2}} e^{qN} f \right\|_{L^p(\mu)} \\ &\leq C(q,k) \left\| e^{qN} f \right\|_{L^2(\mu)} \\ &= C(q,k) \|f\|_q \end{aligned}$$

for a constant $C(q,k)$. We conclude from the last bound that $\mathcal{G}$ can be continuously embedded into $\mathbb{D}_\infty$.

For later use let us also introduce a space closely related to $\mathcal{G}$, that is the Fréchet space $\mathcal{C}$ with norms given by

$$\|f\|_{\mathcal{C}_q}^2 = \left\| e^{q\sqrt{N}} f \right\|_{L^2(\mu)}^2, \; q \geq 0 \tag{24}$$

Thus

$$\mathcal{G} \hookrightarrow \mathcal{C}.$$

Since one can also prove that

$$(\mathcal{S}) \hookrightarrow \mathcal{G}$$

(see [PT]) we get the following chain of continuous inclusions

$$(\mathcal{S}) \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{C} \hookrightarrow L^2(\mu) \hookrightarrow \mathcal{C}^* \hookrightarrow \mathcal{G}^* \hookrightarrow (\mathcal{S})^*. \tag{25}$$

We mention that $\mathcal{G}^*$ forms a topological subalgebra of $(\mathcal{S})^*$ with respect to the Wick product.

3 Approach and results

In this Section we want to present an approach to study strong solutions of stochastic equations for a broader class of driving noises. We shall explain the main principles of our method on the basis of a Brownian motion with (functional) drift, that is the SDE

$$dX_t = b(t, X.)dt + dB_t, \; 0 \leq t \leq T, \; X_0 = x \in \mathbb{R}^d. \tag{26}$$

In Section 5 we will discuss other applications of our technique.

general approach
↓

additive processes like Lévy processes
↗
solutions of stochastic equations → broader class of driving processes
↘
fractional Brownian motion, fractional Lévy processes

In the paper [P2] we demonstrate how our method can be used to capture stochastic equations with additive driving noise. Let us mention that while additive processes are strong

Markov processes, the fractional Brownian motion or more general the fractional Lévy process are stochastic processes, which are in general not Markovian. The latter processes may even not enjoy the semimartingale property. For more information about additive processes resp. fractional Lévy processes we refer to [Be], [JS], [Sa] resp. [DÜ], [DS].

For notational convenience we will from now on suppress the initial value x of (26) in formulas by setting $x = 0$. In order to avoid explicit summations of Cartesian components in multi-dimensional stochastic integrals we will occasionally use the abbreviation

$$\int_0^t \varphi(s,\omega)dB_s = \sum_{j=1}^d \int_0^t \varphi^{(j)}(s,\omega)dB_s^{(j)}.$$

In the sequel we consider the filtered probability space

$$(\Omega, \mathcal{F}, \mu), \{\mathcal{F}_t\}_{t\geq 0}, \tag{27}$$

where $(\Omega, \mathcal{F}, \mu)$ is the white noise space (9) and where $\{\mathcal{F}_t\}_{t\geq 0}$ is the $\mu-$augmented filtration generated by B_t.

We want to motivate the forthcoming considerations by the following observation made in [LP].

Proposition 1 *Let the drift coefficient $b : [0,T]\times\mathcal{W}^d \longrightarrow \mathbb{R}^d$ in (26) be bounded and Lipschitz continuous. Then there exists a unique strong solution X_t of (26), which can be explicitly represented as*

$$\varphi\left(t, X_\cdot^{(i)}(\omega)\right) = E_{\widehat{\mu}}\left[\varphi\left(t, \widehat{B}_\cdot^{(i)}(\widehat{\omega})\right)\mathcal{E}_T^{\diamond}(b)\right] \tag{28}$$

for all $\varphi : [0,T]\times\mathcal{W}\longrightarrow\mathbb{R}$ such that $\varphi\left(t, B_\cdot^{(i)}\right)\in L^2(\mu)$ for all $0\leq t\leq T$, $i = 1,...,d$, where $\mathcal{E}_T^{\diamond}(b)$ is defined as

$$\begin{aligned} &\mathcal{E}_T^{\diamond}(b)(\omega,\widehat{\omega}) \\ = \ &\exp^{\diamond}\left(\sum_{j=1}^d\int_0^T\left(W_s^{(j)}(\omega) + b^{(j)}(s,\widehat{B}_\cdot(\widehat{\omega}))\right)d\widehat{B}_s^{(j)}(\widehat{\omega})\right. \\ &\left. -\frac{1}{2}\int_0^T\left(W_s^{(j)}(\omega) + b^{(j)}(s,\widehat{B}_\cdot(\widehat{\omega}))\right)^{\diamond 2}ds\right). \end{aligned} \tag{29}$$

Here $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mu}\right), \left(\widehat{B}_t\right)_{t\geq 0}$ is a copy of the quadruple $(\Omega,\mathcal{F},\mu)$, $(B_t)_{t\geq 0}$ in (9). The symbol $E_{\widehat{\mu}}$ stands for a Pettis integral of random elements $\Phi : \widehat{\Omega}\longrightarrow(\mathcal{S})^*$ with respect to the measure $\widehat{\mu}$. Further $W_t^{(j)}$ in the Wick exponential of (29)- where the Wick product $\diamond$ is taken with respect μ- denotes the white noise of $B_t^{(j)}$ in the Hida space $(\mathcal{S})^*$ (see (15)). The stochastic integrals $\int_0^T\phi(t,\omega)d\widehat{B}_s^{(j)}(\widehat{\omega})$ in (29) are defined for predictable integrands $\phi(t,\omega)$ taking values in the conuclear space $(\mathcal{S})^*$. See [KX] for definitions. The other integral type turning up in (29) is in the sense of Pettis.

Remark 2

(i) *For a sequence of partitions* $0 = t_1^n < t_2^n < ... < t_{m_n}^n = T$ *of the interval* $[0, T]$ *with* $\max_{i=1}^{m_n - 1} \left| t_{i+1}^n - t_i^n \right| \longrightarrow 0$ *the stochastic integral of the white noise* $W^{(j)}$ *can be written as*

$$\int_0^T W_s^{(j)}(\omega) d\widehat{B}_s^{(j)}(\widehat{\omega}) = \lim_{n \longrightarrow \infty} \sum_{i=1}^{m_n} (\widehat{B}_{t_{i+1}^n}^{(j)}(\widehat{\omega}) - \widehat{B}_{t_i^n}^{(j)}(\widehat{\omega})) W_{t_i^n}^{(j)}(\omega)$$

in $L^2(\lambda \times \widehat{\mu}; (\mathcal{S})^*)$*. For more information about stochastic integration on conuclear spaces we refer to [KX].*

(ii) *The integrand under the expectation* $E_{\widehat{\mu}}$ *in (28) is even Bochner integrable. See [LP].*

For the sake of completeness we give a proof of Proposition 1, which however slightly deviates from the one in *[LP]*. For this purpose we need

Lemma 3 *Let* $(M, \mathcal{B}, m)$ *be a measure space. Suppose a function* $\Phi : M \longrightarrow (\mathcal{S})^*$ *satisfies*

$$S(\Phi(\cdot))(\phi)$$

is measurable for all $\phi \in (\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d$*. Further, denoting by* $(|\cdot|_p)_{p \geq 0}$ *the family of increasing compatible seminorms of* $(\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d$ *we assume that there exist* K*,* a*,* $p \geq 0$ *such that*

$$\int_M |S(\Phi(u))(\phi)| \, m(du) \leq K \exp(a \, |\phi|_p^2)$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d$*. Then* Φ *is Pettis integrable and for any* $E \in \mathcal{B}$ *we have that*

$$S\left(\int_E \Phi(u) m(du)\right)(\phi) = \int_E S(\Phi(u))(\phi) m(du)$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d$*.*

Proof. See e.g. [Ku, Theorem 13.4]. ■

We are coming to the proof of Proposition 1.

Proof of Proposition 1. Without loss of generality we provide the proof for $d = 1$ and bounded functions φ. By assumption the SDE (26) has a unique strong solution $X_t \in L^2(\mu)$. So applying the $S-$transform to X_t we obtain that

$$S(\varphi(t, X.))(\phi) = E_\mu \left[\varphi(t, X.(\omega + \phi)) \right] \tag{30}$$

for all $\phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. Then in virtue of the Girsanov theorem the stochastic process $Y_t(\omega) = X_t(\omega + \phi)$ is a solution of the SDE

$$dY_t = b(t, Y.) + \phi(t) dt + dB_t, \;\; X_0 = x, 0 \leq t \leq T.$$

Using Girsanov's change of measure in (30) (repeatedly) we find that

$$S(X_t)(\phi) = E_{\widehat{\mu}}\left[\varphi\left(t, \widehat{B}_{\cdot}\right)\mathcal{E}(M_t^{\phi})\right]$$

for all $\phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$, where $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu}\right), \left(\widehat{B}_t\right)_{t\geq 0}$ is a copy of the quadruple $(\Omega, \mathcal{F}, \mu)$, $(B_t)_{t\geq 0}$ and where $\mathcal{E}(M_t^{\phi})$ denotes the Doleans-Dade exponential for the martingale

$$M_t^{\phi}(\widehat{\omega}) = \int_0^T \left(b(t, \widehat{B}_{\cdot}(\widehat{\omega})) + \phi(t)\right) d\widehat{B}_t(\widehat{\omega}),$$

that is

$$\begin{aligned}
&\mathcal{E}(M_t^{\phi}) \\
= \; &\exp\left(\int_0^T \left(b(t, \widehat{B}_{\cdot}) + \phi(t)\right) d\widehat{B}_t - \frac{1}{2}\int_0^T \left(b(t, \widehat{B}_{\cdot}) + \phi(t)\right)^2 dt\right)
\end{aligned}$$

We know from (17) that

$$S(W_t)(\phi) = \phi(t)$$

for all $\phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. Then by appealing to the definition of Wick exponentials (21), Remark 2 and the properties of the S-transform (see (18)) we see that

$$S(\Phi(\widehat{\omega}, \cdot))(\phi) = \varphi\left(t, \widehat{B}_{\cdot}(\widehat{\omega})\right)\mathcal{E}(M_t^{\phi})(\widehat{\omega}),$$

where the map $\Phi : \Omega \times \widehat{\Omega} \longrightarrow (\mathcal{S})^*$ is given by

$$\Phi(\widehat{\omega}, \omega) = \varphi\left(\widehat{B}_t(\widehat{\omega})\right)\mathcal{E}_T^{\diamond}(b)(\omega, \widehat{\omega})$$

with $\mathcal{E}_T^{\diamond}(b)$ as in (29). It is clear that $S(\Phi(\widehat{\omega}, \cdot))(\phi)$ is $\widehat{\omega}$-measurable for all ϕ. Further invoking Hölder's inequality and the supermartingale property of Doleans-Dade exponentials we get the estimate

$$\begin{aligned}
&E_{\widehat{\mu}}\left[\left|S(\Phi(\widehat{\omega}, \cdot))(\phi)\right|\right] \\
= \; &E_{\widehat{\mu}}\left[\left|\varphi\left(\widehat{B}_t\right)\mathcal{E}(M_t^{\phi})\right|\right] \\
\leq \; &K \cdot E_{\widehat{\mu}}^{\frac{1}{2}}\left[\mathcal{E}\left(\int_0^T 2\left(b(t, \widehat{B}_{\cdot}) + \operatorname{Re}\phi(t)\right) d\widehat{B}_t\right)\right]\exp(a\int_0^T |\phi(t)|^2\, dt) \\
\leq \; &K\exp(a\,|\phi|_0^2),
\end{aligned}$$

where a, $K \geq 0$ are constants and $|\phi|_0 = \|\phi\|_{L^2(\mathbb{C})}$. Then using Lemma 3 we find

$$S(X_t)(\phi) = S(E_{\widehat{\mu}}[\Phi])(\phi)$$

for all ϕ. The result follows from the injectivity of the $S-$transform. ∎

Adopting the notation in [KS] we want to define a certain class of progressively measurable functionals. Given real separable Banach spaces E_1 and E_2 we shall denote by $C_{\uparrow}^{\infty}(E_1; E_2)$ the space of continuous maps $F : E_1 \longrightarrow E_2$ such that for all $n \in \mathbb{N}$ and $e_0^1, ..., e_n^1 \in E_1$ with

$$(y_1, ..., y_n) \longmapsto F\left(e_0^1 + \sum_{j=1}^{n} y_j e_j^1\right)$$

belongs to $C^{\infty}(\mathbb{R}^n; E_2)$ there exists a continuous map $F^{(n)}$ from E_1 into the space of continuous multilinear maps $L\left(\times_{i=1}^{n} E_1; E_2\right)$ such that

$$\frac{\partial^n F}{\partial y_1 ... \partial y_n}\left(e_0^1 + \sum_{j=1}^{n} y_j e_j^1\right)\Bigg|_{y_1 = ... = y_n = 0} = F^{(n)}(e_0^1)(e_1^1, ..., e_n^1)$$

and

$$\left\| F^{(n)}(e_0^1) \right\|_{L\left(\times_{i=1}^{n} E_1; E_2\right)} \leq C_n \left(1 + \left\| e_0^1 \right\|_{E_1}\right)^{\gamma_n}$$

for some C_n, $\gamma_n < \infty$.

Definition 4 *A measurable functional* $F : [0, \infty) \times C([0, \infty); E_1) \longrightarrow E_2$ *is said to be a* smooth, tempered, non-anticipating function, *if for all* $T \geq 0$ *there exists a function* $F(T) \in C_{\uparrow}^{\infty}(C([0, T]; E_1); E_2)$ *such that*

$$F(T, \phi) = F(T)(\phi|_{[0,T]})$$

for all $\phi \in C([0, \infty), E_1)$ *and such that for all* $T > 0$, $n \in \mathbb{N}_0, 0 \leq t \leq T$ *and* $\phi \in C([0, t]; E_1)$ *:*

$$\left\| F(t)^{(n)}(\phi) \right\|_{L\left(\times_{i=1}^{n} C([0,t]; E_1); E_2\right)} \leq C_n(T) \left(1 + \left\| e_0^1 \right\|_{E_1}\right)^{\gamma_n(T)}$$

for some $C_n(T)$, $\gamma_n(T) < \infty$.

Remark 5 *Let the drift coefficient* b *in (26) be a smooth, tempered, non-anticipating function. Then Lemma 2.9 in [KS] shows that the solution of (26) belongs to the domain of the number operator* N. *In particular the solution is Malliavin differentiable.*

For later use we need to define the norm

$$\|f\|_{\mathcal{N}^q} = \|N^q f\|_{L^2(\mu; \mathbb{R}^d)} \tag{31}$$

and its dual norm given by

$$\|F\|_{\mathcal{N}^{-q}} = \left\| N^{-q} F \right\|_{L^2(\mu; \mathbb{R}^d)}, \tag{32}$$

where N is the number operator and $f \in Dom(N^q)$ and $F \in Dom(N^{-q})$, $q \geq 0$.

For a moment let b be as in Proposition 1. Then the strong solution X_t of (26) takes the explicit form

$$X_t^{(i)} = E_{\widehat{\mu}}\left[\widehat{B}_t^{(i)} \mathcal{E}_T^{\diamond}(b)\right],$$

where $\mathcal{E}_T^{\diamond}(b)$ is given by (29). The latter gives rise to guess that the expression on the right hand side of the equation, that is

$$Y_t^b = \left(Y_t^{1,b}, ..., Y_t^{d,b}\right) \tag{33}$$

with coordinates

$$Y_t^{i,b} \overset{def}{=} E_{\widehat{\mu}}\left[\widehat{B}_t^{(i)} \mathcal{E}_T^{\diamond}(b)\right]$$

for $i = 1, ..., d$ will still solve (26), if b is replaced by a measurable function fulfilling certain integrability conditions.

By using an approximation argument we will show that the object Y_t^b defined by (33) is a strong solution of (26). In doing so we will resort to the following result:

Theorem 6 *Suppose there exists a sequence of progressively measurable functionals b_n : $[0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ with $b_0 = b$, which fulfills (2), (3) for $n \geq 1$ and the integrability condition*

$$\sup_{n \geq 0} E_\mu \left[\exp(512 \int_0^T \|b_n(s, B.)\|^2 \, ds)\right] < \infty. \tag{34}$$

Assume that $b_n, n \geq 1$ in (26) admit Malliavin differentiable solutions $X_t^{(n)} = Y_t^{b_n}$ of (26) (see Remark 5). Further, setting

$$R_n := E_\mu^{\frac{1}{2}}\left[J_n\right] \tag{35}$$

with

$$J_n := \sum_{j=1}^d \left(2 \int_0^T \left(b_n^{(j)}(s, B.) - b^{(j)}(s, B.)\right)^2 ds + \left(\int_0^T \left|b^{(j)}(s, B.)^2 - b_n^{(j)}(s, B.)^2\right| ds\right)^2\right) \tag{36}$$

we require that the factor R_n tends to zero, that is

$$R_n \longrightarrow 0 \text{ for } n \longrightarrow \infty. \tag{37}$$

Then for all $i = 1, ..., d$

$$Y_t^{i,b} \in L^2(\mu).$$

and the "weight"

$$L(n,m) := \left\|Y_t^{i,b_n} - Y_t^{i,b_m}\right\|_{\mathcal{N}^{-\frac{3}{2}}}^{\frac{1}{2}}$$

converges to zero for $n, m \longrightarrow \infty$.
Moreover if

$$\underline{\lim}_{n,m \longrightarrow \infty} L(n,m) \left\|Y_t^{i,b_n} - Y_t^{i,b_m}\right\|_{\mathcal{N}^{\frac{1}{2}}}^{\frac{3}{2}} = 0, \tag{38}$$

with norms $\|\cdot\|_{\mathcal{N}^{\frac{1}{2}}}$ and $\|\cdot\|_{\mathcal{N}^{-\frac{3}{2}}}$ as in (31), (32) then

$$Y_t^{i,b_n} - Y_t^{i,b} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

in $L^2(\mu)$.

In persuing our aim to verify Y_t^bas a strong solution of (26) Theorem 6 will play an essential rôle. The proof of this statement calls for a series of auxiliary results. Under the assumptions of Theorem 6 we will e.g. successively show that

1. Y_t^b is in the Hida distribution space $(\mathcal{S})^*$ (Lemma 7),
2. $Y_t^b \in L^2(\mu)$ (Lemma 9).

The first Lemma provides a condition under which the process Y_t^b is a well-defined object in the Hida distribution space.

Lemma 7 *Assume that*

$$E_\mu \left[\exp \left(36 \int_0^T \|b(s, B.)\|^2 \, ds \right) \right] < \infty, \tag{39}$$

where the drift $b : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ is measurable. Then the coordinates of the process Y_t^b, defined in (33), that is

$$Y_t^{i,b} = E_{\widehat{\mu}} \left[\widehat{B}_t^{(i)} \mathcal{E}_T^{\diamond}(b) \right] \tag{40}$$

are elements of the Hida distribution space.

Proof. Without loss of generality we give the proof for the case $d = 1$. Set $\Phi(\widehat{\omega}, \omega) = \varphi\left(\widehat{B}_t(\widehat{\omega})\right) \mathcal{E}_T^{\diamond}(b)(\omega, \widehat{\omega})$. Then by assumption, Hölder's inequality and the supermartingale property of Doleans-Dade exponentials we get the upper bound

$$\begin{aligned}
& E_{\widehat{\mu}} \left[|S(\Phi(\widehat{\omega}, \cdot))(\phi)| \right] \\
= \;& E_{\widehat{\mu}} \left[\left| \varphi\left(\widehat{B}.\right) \mathcal{E}(M_t^{\phi}) \right| \right] \\
\leq \;& const. \cdot E_{\widehat{\mu}}^{\frac{1}{4}} \left[\mathcal{E}\left(\int_0^T 4 \left(b(t, \widehat{B}.) + \operatorname{Re} \phi(t) \right) d\widehat{B}_t \right) \right] \\
& \cdot E_{\widehat{\mu}}^{\frac{1}{4}} \left[\exp\left(\int_0^T 8 \left(b(t, \widehat{B}.) + \operatorname{Re} \phi(t) \right)^2 dt \right.\right. \\
& \left.\left. + \int_0^T 2b^2(t, \widehat{B}.) dt + \int_0^T 4 \left| b(t, \widehat{B}.) \right| |\phi(t)| \, dt + \int_0^T 2 |\phi(t)|^2 \, dt \right) \right] \\
\leq \;& const. E_{\widehat{\mu}}^{\frac{1}{4}} \left[\exp(36 \int_0^T b^2(t, \widehat{B}.) dt) \right] \exp(9 \|\phi\|_{L^2(\mathbb{R};\mathbb{C})}^2) \\
\leq \;& const. \exp(9 |\phi|_0^2)
\end{aligned}$$

for all $\phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. So applying Lemma 3 yields the result. ■

Lemma 8 *Let $b_n : [0,T] \times \mathcal{W}^d \rightarrow \mathbb{R}^d$ be a sequence of progressively measurable functionals with $b_0 = b$ such that the integrability condition (34) holds.Then*

$$\begin{aligned}
& \left| S(Y_t^{i,b_n} - Y_t^{i,b})(\xi) \right| \\
\leq \;& const. R_n \exp(34 \int_0^T \|\xi(s)\|^2 \, ds)
\end{aligned} \tag{41}$$

for all $\xi \in (S_{\mathbb{C}}(\mathbb{R}))^d$, $i = 1, ..., d$ with the factor R_n as in (35).

Proof. For $i = 1, ..., d$ we find by Proposition 1 and (17) that

$$\begin{aligned}
&\left|S(Y_t^{i,b_n} - Y_t^{i,b})(\xi)\right| \\
\leq\; & E_{\widehat{\mu}}\left[\left(\left|\widehat{B}_t^{(i)}\right| \exp(\sum_{j=1}^d \mathrm{Re}(\int_0^T (b^{(j)}(s,\widehat{B}.) + \xi^{(j)}(s))d\widehat{B}_s^{(j)} - \frac{1}{2}\int_0^T (b^{(j)}(s,\widehat{B}.) + \xi^{(j)}(s))^2 ds))\right)\right. \\
&\cdot\left|\exp(\sum_{j=1}^d \int_0^T (b_n^{(j)}(s,\widehat{B}.) - b^{(j)}(s,\widehat{B}.))d\widehat{B}_s^{(j)} + \frac{1}{2}\int_0^T (b^{(j)}(s,\widehat{B}.)^2 - b_n^{(j)}(s,\widehat{B}.)^2)ds)\right. \\
&\left.\left. + \int_0^T \xi^{(j)}(s)(b^{(j)}(s,\widehat{B}.) - b_n^{(j)}(s,\widehat{B}.))ds) - 1\right|\right]
\end{aligned}$$

Since

$$|\exp(z) - 1| \leq |z| \exp(|z|)$$

it follows with the help of Hölder's inequality that

$$\begin{aligned}
&\left|S(Y_t^{i,b_n} - Y_t^{i,b})(\xi)\right| \leq E_{\widehat{\mu}}^{\frac{1}{2}}\left[|Q_n|^2\right] \\
&\cdot E_{\widehat{\mu}}^{\frac{1}{2}}\left[\left(\left|\widehat{B}_t^{(i)}\right| \exp(\sum_{j=1}^d \mathrm{Re}(\int_0^T (b^{(j)}(s,\widehat{B}.) + \xi^{(j)}(s))d\widehat{B}_s - \frac{1}{2}\int_0^T (b^{(j)}(s,\widehat{B}.) + \xi^{(j)}(s))^2 ds))\right)^2\right. \\
&\left.\exp(2\,|Q_n|)\right]
\end{aligned}$$

where

$$\begin{aligned}
&Q_n \\
=\; & \sum_{j=1}^d \int_0^T (b_n^{(j)}(s,\widehat{B}.) - b^{(j)}(s,\widehat{B}.))d\widehat{B}_s^{(j)} + \frac{1}{2}\int_0^T (b^{(j)}(s,\widehat{B}.)^2 - b_n^{(j)}(s,\widehat{B}.)^2)ds \\
& + \int_0^T \xi^{(j)}(s)(b^{(j)}(s,\widehat{B}.) - b_n^{(j)}(s,\widehat{B}.))ds.
\end{aligned}$$

We have that

$$\begin{aligned}
& E_{\widehat{\mu}}\left[|Q_n|^2\right] \leq 9d^2 \exp(\int_0^T \|\xi(s)\|^2\, ds) \\
&\cdot E_{\widehat{\mu}}\left[\sum_{j=1}^d \left\{\left(\int_0^T (b_n^{(j)}(s,\widehat{B}.) - b^{(j)}(s,\widehat{B}.))d\widehat{B}_s^{(j)}\right)^2 + \left(\int_0^T (b^{(j)}(s,\widehat{B}.)^2 - b_n^{(j)}(s,\widehat{B}.)^2)ds\right)^2\right.\right. \\
&\left.\left. + \int_0^T (b^{(j)}(s,\widehat{B}.) - b_n^{(j)}(s,\widehat{B}.))^2 ds\right\}\right] \\
=\; & 3\exp(\int_0^T \|\xi(s)\|^2\, ds) E_{\widehat{\mu}}\left[J_n\right],
\end{aligned}$$

where

$$J_n = \sum_{j=1}^{d} 2\int_0^T \left(b_n^{(j)}(s,\widehat{B}.) - b^{(j)}(s,\widehat{B}.)\right)^2 ds + \left(\int_0^T \left|b^{(j)}(s,\widehat{B}.)^2 - b_n^{(j)}(s,\widehat{B}.)^2\right| ds\right)^2$$

Further we get that

$$\begin{aligned}
& E_{\widehat{\mu}}\left[\left(\left|\widehat{B}_t^{(i)}\right| \exp(\sum_{j=1}^{d} \mathrm{Re}(\int_0^T (b^{(j)}(s,\widehat{B}.) + \xi^{(j)}(s))d\widehat{B}_s^{(j)} - \frac{1}{2}\int_0^T (b^{(j)}(s,\widehat{B}.) + \xi^{(j)}(s))^2 ds))\right)^2 \right. \\
& \left. \exp(2\left|Q_n\right|)\right] \\
\leq\; & E_{\widehat{\mu}}^{\frac{1}{2}}\left[\left(\left|\widehat{B}_t^{(i)}\right| \exp(\sum_{j=1}^{d} \mathrm{Re}(\int_0^T (b^{(j)}(s,\widehat{B}.) + \xi^{(j)}(s))d\widehat{B}_s^{(j)} - \frac{1}{2}\int_0^T (b^{(j)}(s,\widehat{B}.) + \xi^{(j)}(s))^2 ds))\right)^4\right] \\
& \cdot\frac{1}{\sqrt{2}}\left(E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(-8\,\mathrm{Re}\,Q_n)\right] + E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(8\,\mathrm{Re}\,Q_n)\right]\right. \\
& \left. + E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(-8\,\mathrm{Im}\,Q_n)\right] + E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(8\,\mathrm{Im}\,Q_n)\right]\right).
\end{aligned}$$

By Hölder's inequality again and the supermartingale property of Doléans-Dade exponentials we obtain the estimate

$$\begin{aligned}
& E_{\widehat{\mu}}\left[\exp(-8\,\mathrm{Re}\,Q_n)\right] \\
\leq\; & E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(\sum_{j=1}^{d} 128\int_0^T (b^{(j)}(s,\widehat{B}.) - b_n^{(j)}(s,\widehat{B}.))^2 ds - 8\int_0^T (b^{(j)}(s,\widehat{B}.)^2 - b_n^{(j)}(s,\widehat{B}.)^2) ds\right. \\
& \left. +8(\int_0^T (\mathrm{Re}(\xi^{(j)}(s)))^2 ds + \int_0^T (b^{(j)}(s,\widehat{B}.) - b_n^{(j)}(s,\widehat{B}.))^2 ds)\right] \\
\leq\; & L_n \exp(4\int_0^T \left\|\xi(s)\right\|^2 ds),
\end{aligned}$$

where

$$\begin{aligned}
& L_n \\
=\; & E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(\sum_{j=1}^{d} 128\int_0^T (b^{(j)}(s,\widehat{B}.) - b_n^{(j)}(s,\widehat{B}.))^2 ds + 8\int_0^T \left|b^{(j)}(s,\widehat{B}.)^2 - b_n^{(j)}(s,\widehat{B}.)^2\right| ds)\right].
\end{aligned}$$

Similarly we deduce that

$$\begin{aligned}
& E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(8\,\mathrm{Re}\,Q_n)\right] \\
\leq\; & L_n \exp(4\int_0^T \left\|\xi(s)\right\|^2 ds).
\end{aligned}$$

Also $E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(-8\,\mathrm{Im}\,Q_n)\right]$ and $E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp(8\,\mathrm{Im}\,Q_n)\right]$ have the same upper bound as in the previous inequality.

Finally we find

$$
\begin{aligned}
& E_{\widehat{\mu}}^{\frac{1}{2}}\left[\left(\left|\widehat{B}_t^{(i)}\right| \exp(\sum_{j=1}^{d} \operatorname{Re}(\int_0^T (b^{(j)}(s,\widehat{B}.)+\xi^{(j)}(s))d\widehat{B}_s^{(j)} - \frac{1}{2}\int_0^T (b^{(j)}(s,\widehat{B}.)+\xi^{(j)}(s))^2 ds))\right)^4\right] \\
\leq \; & E_{\widehat{\mu}}^{\frac{1}{4}}\left[\left(\widehat{B}_t^{(i)}\right)^8\right] E_{\widehat{\mu}}^{\frac{1}{8}}\left[\exp(512\int_0^T \left\|b(s,\widehat{B}.)\right\|^2 ds\right] \exp(64\int_0^T \left\|\xi(s)\right\|^2 ds).
\end{aligned}
$$

Altogether we have shown that

$$
\begin{aligned}
& \left|S(Y_t^{i,b_n} - Y_t^{i,b})(\xi)\right| \\
\leq \; & const.R_n \exp(34\int_0^T \left\|\xi(s)\right\|^2 ds)
\end{aligned}
$$

with R_n as in (35). ■

Lemma 9 *Let $b_n : [0,T] \times \mathcal{W}^d \rightarrow \mathbb{R}^d$ with $b_0 = 0$ be a sequence of progressively measurable functionals satisfying the conditions (34) and (37) . Further impose on b_n, $n \geq 1$ to fulfill (2) and (3). Then the process Y_t^b given by (33) is square integrable for all t.*

Proof. Since (2) is valid for b_n, $n \geq 1$ we conclude from Lemma 7 that $Y_t^{\varphi_n}$ are square integrable unique solutions of the corresponding Brownian motion with drift. Further with the help of Hölder's inequality and the supermartingale property of Doléans-Dade exponentials it follows that

$$
\begin{aligned}
\left\|Y_t^{i,b_n}\right\|_{L^2(\mu)}^2 &= E_{\widehat{\mu}}\left[\left(\widehat{B}_t^{(i)}\right)^2 \mathcal{E}\left(\int_0^T b_n(s,\widehat{B}.)d\widehat{B}_s\right)\right] \\
&\leq const. \sup_{n\geq 1} E_{\widehat{\mu}}\left[\exp\left(6\int_0^T \left\|b_n(s,\widehat{B}.)\right\|^2 ds\right)\right]^{\frac{1}{4}} \leq M < \infty. \qquad (42)
\end{aligned}
$$

Thus the sequence $Y_t^{b_n}$ is relatively compact in $L^2(\mu;\mathbb{R}^d)$ in the weak sense. This implies that there exists a subsequence of $Y_t^{b_n}$ which converges to an element $Z_t \in L^2(\mu;\mathbb{R}^d)$ weakly. Without loss of generality we assume that

$$
Y_t^{b_n} \longrightarrow Z_t \text{ weakly for } n \longrightarrow \infty.
$$

In particular, since

$$
\mathcal{E}\left(\int_{\mathbb{R}} \xi(s)dB_s\right) \in L^p(\mu),\ p > 0,
$$

one gets that

$$
E_\mu\left[Y_t^{i,\varphi_n}\mathcal{E}\left(\int_{\mathbb{R}} \xi(s)dB_s\right)\right] \longrightarrow E_\mu\left[Z_t^{(i)}\mathcal{E}\left(\int_{\mathbb{R}} \xi(s)dB_s\right)\right] \text{ for } n \longrightarrow \infty.
$$

On the other hand the estimate (41) in Lemma 8 gives

$$\begin{aligned}
& E_\mu\left[Y_t^{i,\varphi_n}\mathcal{E}\left(\int_{\mathbb{R}}\xi(s)dB_s\right)\right] = E_{\widehat{\mu}}\left[\widehat{B}_t^{(i)}\mathcal{E}\left(\int_0^T\left(\varphi_n(s,\widehat{B}.)+\xi(s)\right)dB_s\right)\right] \\
\longrightarrow\ & E_{\widehat{\mu}}\left[\widehat{B}_t^{(i)}\mathcal{E}\left(\int_0^T\left(b(s,\widehat{B}.)+\xi(s)\right)dB_s\right)\right] \\
=\ & S(Y_t^b)(\xi),\ \xi\in(\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d.
\end{aligned}$$

Thus

$$S(Y_t^{i,b})(\xi) = S(Z_t^{(i)})(\xi), \xi\in(\mathcal{S}_{\mathbb{C}}(\mathbb{R}))^d.$$

From the injectivity of the S-transform we see that $Y_t = Z_t \in L^2(\mu;\mathbb{R}^d)$. ■

The following results are crucial for our main results (Theorem 17, 18, 19) in this section.

Lemma 10 *Retain the conditions of Lemma 9 for the sequence of progressively measurable functionals $b_n : [0,T]\times\mathcal{W}^d \longrightarrow \mathbb{R}^d$. Then*

$$Y_t^{(b_n)} \overset{\|\cdot\|_{\mathcal{N}^{-\frac{3}{2}}}}{\longrightarrow} Y_t^{(b)} \text{ uniformly in } t \text{ as } n\longrightarrow\infty. \tag{43}$$

Proof. Without loss of generality we give the proof for the case $d=1$. Now we assume that our white noise framework is developed for the space $L^2([0,T])$, that is for the time-interval $[0,T]$ instead of $\mathbb{R}$. Note that such a change does not affect our results. See [DPV].

Denote by $\mathcal{S}([0,T])$ a Schwartz space based on a standard construction with respect to a complete ONS $\{\xi_k\}_{k\geq 1}$ of $L^2([0,T])$. See e.g. [O]. Let $\xi\in\mathcal{S}([0,T])$. Define $G(\xi)=S(F)(\xi)$ for $\xi\in\mathcal{S}([0,T])$ and let $z\in\mathbb{C}$. Then $G(z\xi)$ is an entire analytic function in z. One can show that $G(z\xi)$ has a power expansion

$$G(z\xi)=\sum_{m\geq 0} z^m G^{(m)}(\xi)$$

with

$$G^{(m)}(\xi)=\frac{1}{n!}\left(D_\xi^m G\right)(0),$$

where D_ξ is the Gâteaux derivative in the direction of ξ.

Define the following symmetric $m-$multilinear form $f^{(m)}$ on $\prod_{j=1}^{m}\mathcal{S}([0,T])$:

$$f^{(m)}(\xi_1,...,\xi_m)=\frac{1}{2^m m!}\sum_{\varepsilon}\varepsilon_1\cdot...\cdot\varepsilon_m G^{(m)}(\varepsilon_1\xi_1+...+\varepsilon_m\xi_m), \tag{44}$$

where the sum is taken over all ε with $\varepsilon_i=\pm 1$, $i=1,...,m$. See e.g. [HKPS].

Denote by $\|\cdot\|_{H.S.}$ the Hilbert-Schmidt norm of an operator. Further let $\{\xi_n\}_{n\in\mathbb{N}}$ be a complete ONB of $L^2([0,T])$. Further assume that the Hilbert-Schmidt of $f^{(m)}$, that is

$$\left\|f^{(m)}\right\|_{H.S.}^2=\sum_{j_1,...,j_m\geq 1}\left|f^{(m)}(\xi_{j_1},...,\xi_{j_m})\right|^2$$

is finite. Then we may identify $f^{(m)}$ with the square integrable symmetric kernel f_m in the m-th homogeneous chaos of F and we obtain that

$$\left\| f^{(m)} \right\|_{H.S.} = \| f_m \|^2_{L^2(\mathbb{R}^m;(\mathbb{R}^d)^{\otimes m})} .$$

See [HKPS]. Next we proceed to determine the Hilbert-Schmidt norm of the kernels $f^{(m)}$ of

$$F = Y_t^{b_n} - Y_t^b \in L^2(\mu).$$

We first observe that

$$\begin{aligned} G^{(1)}(\xi) &= E_{\widehat{\mu}} \left[\widehat{B}_t \left\{ \left(\int_0^T \xi(s) d\widehat{B}_s - \int_0^T \xi(s) b_n(s, \widehat{B}.) ds \right) \mathcal{E} \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) \right. \right. \\ &\quad \left. \left. - \left(\int_0^T \xi(s) d\widehat{B}_s - \int_0^T \xi(s) b(s, \widehat{B}.) ds \right) \mathcal{E} \left(\int_0^T b(s, \widehat{B}.) d\widehat{B}_s \right) \right\} \right] \end{aligned}$$

Then representation (44) entails

$$\begin{aligned} &\left\| f^{(1)} \right\|^2_{H.S.} \\ &= \sum_{j \geq 1} E^2_{\widehat{\mu}} \left[\widehat{B}_t \left\{ \left(\int_0^T \xi_j(s) d\widehat{B}_s - \int_0^T \xi_j(s) b_n(s, \widehat{B}.) ds \right) \mathcal{E} \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) \right. \right. \\ &\quad \left. \left. - \left(\int_0^T \xi_j(s) d\widehat{B}_s - \int_0^T \xi_j(s) b(s, \widehat{B}.) ds \right) \mathcal{E} \left(\int_0^T b(s, \widehat{B}.) d\widehat{B}_s \right) \right\} \right] \end{aligned}$$

The case $m = 2$ yields

$$\begin{aligned} &\left\| f^{(2)} \right\|^2_{H.S.} \\ &= \sum_{j_1, j_2 \geq 1} \frac{1}{4} E^2_{\widehat{\mu}} \left[\widehat{B}_t \left\{ - \left(\int_0^T \xi_{j_1}(s) \xi_{j_2}(s) ds \right) \mathcal{E} \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) \right. \right. \\ &\quad + \left(\int_0^T \xi_{j_1}(s) d\widehat{B}_s - \int_0^T \xi_{j_1}(s) b_n(s, \widehat{B}.) ds \right) \left(\int_0^T \xi_{j_1}(s) d\widehat{B}_s - \int_0^T \xi_{j_1}(s) b_n(s, \widehat{B}.) ds \right) \\ &\quad \mathcal{E} \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) \\ &\quad + \left(\int_0^T \xi_{j_1}(s) \xi_{j_2}(s) ds \right) \mathcal{E} \left(\int_0^T b(s, \widehat{B}.) d\widehat{B}_s \right) \\ &\quad - \left(\int_0^T \xi_{j_1}(s) d\widehat{B}_s - \int_0^T \xi_{j_1}(s) b(s, \widehat{B}.) ds \right) \left(\int_0^T \xi_{j_1}(s) d\widehat{B}_s - \int_0^T \xi_{j_1}(s) b(s, \widehat{B}.) ds \right) \\ &\quad \left. \left. \mathcal{E} \left(\int_0^T b(s, \widehat{B}.) d\widehat{B}_s \right) \right\} \right] \end{aligned}$$

Denote by $I_n(f)$ the $n-$th homogeneous chaos of a symmetric function f for $n \geq 1$. Then integration by parts supplies

$$I_2 \left(\xi_{j_1} \widehat{\otimes} \xi_{j_2} \right) = \int_0^T \xi_{j_1}(s) d\widehat{B}_s \cdot \int_0^T \xi_{j_2}(s) d\widehat{B}_s - \int_0^T \xi_{j_1}(s) \xi_{j_2}(s) ds.$$

The latter implies

$$\begin{aligned}
&\left\|f^{(2)}\right\|_{H.S.}^{2} \\
=&\sum_{j_1,j_2\geq 1}\frac{1}{4}E_{\widehat{\mu}}^{2}\left[\sum_{\mathcal{J}\in 2^{\{1,2\}}}\widehat{B}_t\left\{I_{|\mathcal{J}|}\left(\widehat{\otimes}_{i\in\mathcal{J}}(\xi_{j_i})\right)\right.\right. \\
&\prod_{i\in\{1,2\}\backslash\mathcal{J}}\left(-\int_0^T\xi_{j_i}(s)b_n(s,\widehat{B}.)ds\right)\mathcal{E}\left(\int_0^T b_n(s,\widehat{B}.)d\widehat{B}_s\right) \\
&\left.\left.-I_{|\mathcal{J}|}\left(\widehat{\otimes}_{i\in\mathcal{J}}(\xi_{j_i})\right)\prod_{i\in\{1,2\}\backslash\mathcal{J}}\left(-\int_0^T\xi_{j_i}(s)b(s,\widehat{B}.)ds\right)\mathcal{E}\left(\int_0^T b(s,\widehat{B}.)d\widehat{B}_s\right)\right\}\right],
\end{aligned}$$

where 2^M is the power set and $|\mathcal{J}|$ the cardinality of $\mathcal{J}$.

More generally, using integration by parts applied to finite products of $\int_0^T\xi_{j_i}(s)d\widehat{B}_s$ and $\int_0^T\xi_{j_i}(s)\xi_{j_l}(s)ds$ we deduce by induction that

$$\begin{aligned}
&\left\|f^{(m)}\right\|_{H.S.}^{2} \\
=&\sum_{j_1,\ldots,j_m\geq 1}\frac{1}{(m!)^2}E_{\widehat{\mu}}^{2}\left[\sum_{\mathcal{J}\in 2^{\{1,\ldots,m\}}}\widehat{B}_t\left\{I_{|\mathcal{J}|}\left(\widehat{\otimes}_{i\in\mathcal{J}}(\xi_{j_i})\right)\right.\right. \\
&\prod_{i\in\{1,\ldots,m\}\backslash\mathcal{J}}\left(-\int_0^T\xi_{j_i}(s)b_n(s,\widehat{B}.)ds\right)\mathcal{E}\left(\int_0^T b_n(s,\widehat{B}.)d\widehat{B}_s\right) \\
&\left.\left.-I_{|\mathcal{J}|}\left(\widehat{\otimes}_{i\in\mathcal{J}}(\xi_{j_i})\right)\prod_{i\in\{1,\ldots,m\}\backslash\mathcal{J}}\left(-\int_0^T\xi_{j_i}(s)b(s,\widehat{B}.)ds\right)\mathcal{E}\left(\int_0^T b(s,\widehat{B}.)d\widehat{B}_s\right)\right\}\right] \quad (45)
\end{aligned}$$

for all $m\geq 1$, where $I_{|\{\varnothing\}|}=1$ and $\prod_{i\in\{\varnothing\}}=1$ by convention.

By (45) we derive the estimate

$$\begin{aligned}
&\left\|f^{(m)}\right\|_{H.S.}^{2} \\
\leq&\frac{1}{(m!)^2}2^{2m}\sum_{\mathcal{J}\in 2^{\{1,\ldots,m\}}}\sum_{j_1,\ldots,j_m\geq 1}E_{\widehat{\mu}}^{2}\left[\widehat{B}_t\left\{I_{|\mathcal{J}|}\left(\widehat{\otimes}_{i\in\mathcal{J}}(\xi_{j_i})\right)\right.\right. \\
&\prod_{i\in\{1,\ldots,m\}\backslash\mathcal{J}}\left(-\int_0^T\xi_{j_i}(s)b_n(s,\widehat{B}.)ds\right)\mathcal{E}\left(\int_0^T b_n(s,\widehat{B}.)d\widehat{B}_s\right) \\
&\left.\left.-I_{|\mathcal{J}|}\left(\widehat{\otimes}_{i\in\mathcal{J}}(\xi_{j_i})\right)\prod_{i\in\{1,\ldots,m\}\backslash\mathcal{J}}\left(-\int_0^T\xi_{j_i}(s)b(s,\widehat{B}.)ds\right)\mathcal{E}\left(\int_0^T b(s,\widehat{B}.)d\widehat{B}_s\right)\right\}\right].
\end{aligned}$$

Further we get

$$
\begin{aligned}
& \left\| f^{(m)} \right\|_{H.S.}^2 \\
\leq\; & \frac{1}{(m!)^2} 2^{2m+1} \sum_{\mathcal{J} \in 2^{\{1,\ldots,m\}}} \sum_{j_1,\ldots,j_m \geq 1} E_{\widehat{\mu}}^2 \Big[\widehat{B}_t \left\{ I_{|\mathcal{J}|} \left(\widehat{\otimes}_{i \in \mathcal{J}} (\xi_{j_i}) \right) \right. \\
& \left(\prod_{i \in \{1,\ldots,m\} \setminus \mathcal{J}} \left(- \int_0^T \xi_{j_i}(s) b_n(s, \widehat{B}.) ds \right) - \prod_{i \in \{1,\ldots,m\} \setminus \mathcal{J}} \left(- \int_0^T \xi_{j_i}(s) b(s, \widehat{B}.) ds \right) \right) \\
& \left. \mathcal{E} \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) \right\} \Big] + E_{\widehat{\mu}}^2 \Big[\widehat{B}_t \left\{ I_{|\mathcal{J}|} \left(\widehat{\otimes}_{i \in \mathcal{J}} (\xi_{j_i}) \right) \right. \\
& \left. \prod_{i \in \{1,\ldots,m\} \setminus \mathcal{J}} \left(- \int_0^T \xi_{j_i}(s) b(s, \widehat{B}.) ds \right) \left(\mathcal{E} \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) - \mathcal{E} \left(\int_0^T b(s, \widehat{B}.) d\widehat{B}_s \right) \right) \right\} \Big].
\end{aligned}
$$

By Bessel's inequality we conclude that

$$
\begin{aligned}
& \left\| f^{(m)} \right\|_{H.S.}^2 \\
\leq\; & \frac{1}{(m!)^2} 2^{2m+1} \sum_{k=0}^{m} \binom{m}{k} k! \sum_{j_1,\ldots,j_{m-k} \geq 1} E_{\widehat{\mu}} \Big[\widehat{B}_t^2 \\
& \left(\prod_{i=1}^{m-k} \left(- \int_0^T \xi_{j_i}(s) b_n(s, \widehat{B}.) ds \right) - \prod_{i=1}^{m-k} \left(- \int_0^T \xi_{j_i}(s) b(s, \widehat{B}.) ds \right) \right)^2 \\
& \left. \mathcal{E}^2 \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) \right\} \Big] + E_{\widehat{\mu}}^2 \left[\widehat{B}_t^2 \prod_{i=1}^{m-k} \left(- \int_0^T \xi_{j_i}(s) b(s, \widehat{B}.) ds \right)^2 \right. \\
& \left. \left. \left(\mathcal{E} \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) - \mathcal{E} \left(\int_0^T b(s, \widehat{B}.) d\widehat{B}_s \right) \right)^2 \right\} \right].
\end{aligned}
$$

Then applying Parseval's identity gives

$$
\begin{aligned}
& \left\| f^{(m)} \right\|_{H.S.}^2 \\
\leq\; & \frac{1}{(m!)^2} 2^{2m+1} \sum_{k=0}^{m} \binom{m}{k} k! \sum_{\nu=1}^{m-k-1} 2^{2\nu} E_{\widehat{\mu}} \Big[\widehat{B}_t^2 \\
& \int_0^T \left(b_n(s, \widehat{B}.) - b(s, \widehat{B}.) \right)^2 ds \left(\int_0^T b^2(s, \widehat{B}.) ds \right)^{\nu-1} \left(\int_0^T b_n^2(s, \widehat{B}.) ds \right)^{m-k-\nu} \\
& \left. \mathcal{E}^2 \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) \right\} \Big] + E_{\widehat{\mu}}^2 \left[\widehat{B}_t^2 \prod_{i=1}^{m-k} \int_0^T b^2(s, \widehat{B}.) ds \right. \\
& \left. \left. \left(\mathcal{E} \left(\int_0^T b_n(s, \widehat{B}.) d\widehat{B}_s \right) - \mathcal{E} \left(\int_0^T b(s, \widehat{B}.) d\widehat{B}_s \right) \right)^2 \right\} \right].
\end{aligned}
$$

Using

$$|\exp(x)-1| \leq |x|\exp(|x|)$$

Hölder's inequality, the supermartingale property of Doleans-Dade exponentials (see the proof of Lemma 8) we find the bound

$$\begin{aligned}
& E_{\widehat{\mu}}\left[\left(\mathcal{E}\left(\int_0^T b_n(s,\widehat{B}.)d\widehat{B}_s\right)-\mathcal{E}\left(\int_0^T b(s,\widehat{B}.)d\widehat{B}_s\right)\right)^4\right] \\
\leq\ & E_{\widehat{\mu}}^{\frac{1}{2}}\left[|Q_n|^8\right] E_{\widehat{\mu}}^{\frac{1}{2}}\left[\mathcal{E}^8\left(\int_0^T b(s,\widehat{B}.)d\widehat{B}_s\right)\exp(8\,|Q_n|)\right] \\
\leq\ & const.E_{\widehat{\mu}}^{\frac{1}{4}}\left[|Q_n|^2\right],
\end{aligned}$$

where

$$\begin{aligned}
& Q_n \\
=\ & \int_0^T (b_n(s,\widehat{B}.)-b(s,\widehat{B}.))d\widehat{B}_s+\frac{1}{2}\int_0^T (b^2(s,\widehat{B}.)-b_n^2(s,\widehat{B}.))ds.
\end{aligned}$$

Applying Hölder's inequality once more supplies

$$\left\|f^{(m)}\right\|_{H.S.}^2 \leq CR_n,$$

where

$$R_n = E_{\widehat{\mu}}^{\frac{1}{8}}\left[\int_0^T \left(b_n(s,\widehat{B}.)-b(s,\widehat{B}.)\right)^2 ds+\left(\int_0^T (b^2(s,\widehat{B}.)-b_n^2(s,\widehat{B}.))ds\right)^2\right]$$

and where C is a constant being dependent on m. Since $Y_t^{(b_n)}$ is a bounded sequence in $L^2(\mu)$ it follows from (32) that

$$Y_t^{(b_n)} \overset{\|\cdot\|_{\mathcal{N}^{-\frac{3}{2}}}}{\longrightarrow} Y_t^{(b)} \text{ as } n \longrightarrow \infty.$$

■

Using the above auxiliary results we can prove Theorem 6:

Proof of Theorem 6. By using the properties of the number operator we find that

$$\left\|Y_t^{(b_n)}-Y_t^{(b_m)}\right\|_{L^2(\mu;\mathbb{R}^d)}^2 \leq \left\|Y_t^{(b_n)}-Y_t^{(b_m)}\right\|_{\mathcal{N}^{\frac{1}{2}}}^{\frac{3}{2}} \left\|Y_t^{(b_n)}-Y_t^{(b_m)}\right\|_{\mathcal{N}^{-\frac{3}{2}}}^{\frac{1}{2}}$$

for all $m,n \geq 1$. Then by the assumptions of Theorem 6, Lemma 9 and Lemma 10 we obtain the result. ■

We need the following class of approximating functions:

Definition 11 *We denote by $\mathcal{O}$ the class of functions $f:[0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfying (2), (3) and having continuous first order spatial derivatives with compact support in $[0,T]\times\mathbb{R}^d$.*

The next result is a consequence of Theorem 6 and will be later used to construct Malliavin differentiable strong solutions of (26) with irregular coefficients $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$.

Proposition 12 *Let $b_n : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d, n \geq 1$ be a sequence of functions such that the restricted maps $b_n : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, n \geq 1$ belong to the class $\mathcal{O}$. Assume that $b_n : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ vanishes outside of $[0,T] \times \mathbb{R}^d$. Further require that the factor R_n in (35) tends to zero for a Borel measurable $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$. Suppose that*

$$\lim_{r \searrow 0} \sup_{n \geq 1} \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \int_s^{s+r} \int_{\mathbb{R}^d} \left| b_n^{(j)}(t,x) \right|^2 p(t-s;x,y) dx dt = 0 \tag{46}$$

as well as

$$\lim_{r \searrow 0} \sup_{n \geq 1} \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \int_s^{s+r} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_i} b_n^{(j)}(t,x) \right| p(t-s;x,y) dx dt = 0 \tag{47}$$

hold for all $i,j = 1,...,d$, where $p(t;x,y)$ is the $m-$dimensional Gaussian kernel. Then

$$Y_t^{(b_n)} \longrightarrow Y_t^{(b)} \text{ in } L^2(\mu) \text{ uniformly in } t \text{ as } n \longrightarrow \infty.$$

Moreover, $Y_t^{(b)}$ is Malliavin differentiable for all $t \geq 0$.

Proof. Denote by $X_t^{(n)}$ the unique strong solution of

$$X_t^{(n)} = y + \int_0^t b_n(t, X_s^{(n)}) ds + B_t \tag{48}$$

for all $n \geq 1$. Let us first prove that

$$E\left[\left\| D_s X_t^{(n)} \right\|^2 \right] \leq C < \infty \tag{49}$$

for all $n \in \mathbb{N}$, $0 \leq t \leq T$, where C is a constant.

Watanabe's result [W] implies that $X_t^{(n)}$ is Malliavin differentiable for all $t \geq 0$. Thus by applying the Malliavin derivative D_t to (48) we observe that $D_s X_t^{(n)}, 0 \leq s \leq t$ solves the linear equation

$$D_s X_t^{(n)} = \int_s^t \boldsymbol{D}_x b_n(u, X_u^{(n)}) \cdot D_s X_u^{(n)} du + I \tag{50}$$

for all $j = 1,...,d$, where $\boldsymbol{D}_x$ is the ordinary derivative with respect to x and I the identity matrix in $\mathbb{R}^{d \times d}$. Denoting by $\|\cdot\|$ a norm on $\mathbb{R}^{d \times d}$, we get from Gronwall's Lemma that

$$\left\| D_s X_t^{(n)} \right\|^2 \leq const. \exp\left(2 \sum_{i,j=1}^d \int_0^T \left| \frac{\partial}{\partial x_i} b_n^{(j)}(u, X_u^{(n)}) \right| du \right) \text{ a.e.} \tag{51}$$

Then by Hölder's inequality, Girsanov's theorem and (34) we obtain that

$$\begin{aligned}
& E\left[\left\|D_s X_t^{(n)}\right\|^2\right] \\
\leq\ & const.E\left[\exp\left(2\sum_{i,j=1}^{d}\int_0^T\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u,X_u^{(n)})\right|du\right)\right] \\
\leq\ & const.\prod_{i,j=1}^{d}E^{\frac{1}{d^2}}\left[\exp\left(2d^2\int_0^T\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u,X_u^{(n)})du\right|\right)\right] \\
=\ & const.\prod_{i,j=1}^{d}E_{\widehat{\mu}}^{\frac{1}{d^2}}\left[\exp\left(2d^2\int_0^T\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u,\widehat{B}_u+y)\right|du\right)\prod_{j=1}^{d}\mathcal{E}\left(\int_0^T b_n^{(j)}(u,\widehat{B}_u+y)d\widehat{B}_u\right)\right] \\
\leq\ & const.\prod_{i,j=1}^{d}E_{\widehat{\mu}}^{\frac{1}{2d^2}}\left[\exp\left(4d^2\int_0^T\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u,\widehat{B}_u+y)\right|du\right)\right] \\
& \cdot E_{\widehat{\mu}}^{\frac{1}{2}}\left[\prod_{j=1}^{d}\mathcal{E}^2\left(\int_0^T b_n^{(j)}(u,\widehat{B}_u+y)d\widehat{B}_u\right)\right] \\
\leq\ & const.\prod_{i,j=1}^{d}E_{\widehat{\mu}}^{\frac{1}{2d^2}}\left[\exp\left(4d^2\int_0^T\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u,\widehat{B}_u+y)\right|du\right)\right],
\end{aligned} \tag{52}$$

where $\widehat{\mu}$, $\widehat{B}$ are copies of μ, B, respectively. By assumption we know that there exists for all $0<\beta<1$a $\delta>0$ such that

$$\sup_{n\geq 1}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\int_s^{s+\delta}\int_{\mathbb{R}^d}4d^2\left|\frac{\partial}{\partial x_i}b_n^{(j)}(t,x)p(t-s;x,y)\right|dxdt<\beta<1 \tag{53}$$

for all $i,j=1,...,d$.

In bounding above the latter term in (52) we shall invoke an argument of Khas'minskii [Kh]. More precisely, symmetry, Markovianity of $\widehat{B}$, (53), Fubini's theorem and (47) give for all $(s,y) \in [0,T] \times \mathbb{R}^d$ and $i,j = 1,...,d$:

$$\begin{aligned}
& E_{\widehat{\mu}}\left[\exp\left(4d^2\int_s^{s+\delta}\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u,\widehat{B}_{u-s}+y)\right|du\right)\right] \\
\leq \;& 1+\beta+\sum_{m\geq 2}E_{\widehat{\mu}}\left[\frac{1}{m!}4^m d^{2m}\left(\int_s^{s+\delta}\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u,\widehat{B}_{u-s}+y)\right|du\right)^m\right] \\
= \;& 1+\beta+\sum_{m\geq 2}E_{\widehat{\mu}}\left[4^m d^{2m}\int_{s<u_1<...<u_m<s+\delta}\prod_{r=1}^m\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u_r,\widehat{B}_{u_r-s}+y)\right|du_1...du_m\right] \\
\leq \;& 1+\beta+\sum_{m\geq 2}\left(\sup_{n\geq 1}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}E_{\widehat{\mu}}\left[\int_s^{s+\delta}4d^2\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u_r,\widehat{B}_{u_r-s}+y)\right|du\right]\right)^m \\
= \;& 1+\beta+\sum_{m\geq 2}\left(\sup_{n\geq 1}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\int_s^{s+\delta}\int_{\mathbb{R}^d}4d^2\left|\frac{\partial}{\partial x_i}b_n^{(j)}(t,x)p(t-s;x,y)\right|dxdt\right)^m \\
\leq \;& \sum_{m\geq 0}\beta^m = \frac{1}{1-\beta} < \infty. \qquad (54)
\end{aligned}$$

Using the Markov property again yields

$$E_{\widehat{\mu}}\left[\exp\left(4d^2\int_s^{s+q}\left|\frac{\partial}{\partial x_i}b_n^{(j)}(u,\widehat{B}_{u-s}+y)\right|du\right)\right] \leq \left(\frac{1}{1-\beta}\right)^k$$

for all $n \geq 1, (s,y) \in [0,T] \times \mathbb{R}^d$ and q such that $(k-1)\delta \leq q \leq k\delta$ for a $k \in \mathbb{N}_0$.

So it follows from (52) that

$$E\left[\int_0^T \left\|D_s X_t^{(n)}\right\|^2 ds\right] \leq const.T \cdot d^2\left(\frac{1}{1-\beta}\right)^{\frac{k}{2d^2}} < \infty. \qquad (55)$$

for all t. So we obtain together with Meyer's inequalities (see e.g. [N, Theorem 1.5.1]) that

$$\left\|X_t^{(n)}\right\|_{\mathcal{N}^{\frac{1}{2}}} \leq C < \infty$$

for all $n \in \mathbb{N}$ and $0 \leq t \leq T$, where $C < \infty$ is a constant.

In the same way as above Khas'minskii's argument in connection with (46) entails the integrability condition (34). Altogether, applying Lemma 10, Theorem 6 and weak compactness in Hilbert spaces completes the proof. ■

Corollary 13 *Replace condition (47) in Proposition 12 by the requirement*

$$\begin{aligned}
& \lim_{r\searrow 0}\sup_{n\geq 1}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\int_s^{s+r}\int_{\mathbb{R}^{d-1}}\left|\int_{\mathbb{R}}\left\|b_n^{(j)}(t,x_1,...,x_{i-1},\cdot,x_{i+1},...,x_d)\right\|_{BV(-\infty,x_i]}\right. \\
& \left.\frac{\partial}{\partial x_i}p(t-s;x_1,...,x_i,...,x_d,y)dx_i\right|dx_1...dx_{i-1}dx_{i+1}...dx_d dt \\
= \;& 0 \qquad (56)
\end{aligned}$$

for all $i, j = 1, ..., d$, where $\|\cdot\|_{BV(-\infty,x]}$ denotes the bounded variation norm for intervals of the form $[a, x]$.
Then
$$Y_t^{(b_n)} \longrightarrow Y_t^{(b)} \text{ in } L^2(\mu) \text{ uniformly in } t \text{ as } n \longrightarrow \infty.$$
Moreover, $Y_t^{(b)}$ is Malliavin differentiable for all $t \geq 0$.

Proof. Since the coefficients $b_n^{(j)}$ in $\mathcal{O}$ have compact support we can apply integration by parts and get

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \left| \frac{\partial}{\partial x_i} b_n^{(j)}(t, x_1, ..., x_{i-1}, z, x_{i+1}, ..., x_d) \right| p(t-s; x_1, ..., x_{i-1}, z, x_{i+1}, ..., x_d, y) dz \right| \\
= &\left| \int_{\mathbb{R}} \int_{(-\infty, x_i]} \left| \frac{\partial}{\partial x_i} b_n^{(j)}(t, x_1, ..., x_{i-1}, z, x_{i+1}, ..., x_d) \right| dz \frac{\partial}{\partial x_i} p(t-s; x_1, ..., x_i, ..., x_d, y) dx_i \right| \\
= &\left| \int_{\mathbb{R}} \left\| b_n^{(j)}(t, x_1, ..., x_{i-1}, \cdot, x_{i+1}, ..., x_d) \right\|_{BV(-\infty, x_i]} \frac{\partial}{\partial x_i} p(t-s; x_1, ..., x_i, ..., x_d, y) dx_i \right|.
\end{aligned}$$

The latter relation in connection with Proposition 12 yields the proof. ∎

We shall turn our attention to the case of a Brownian motion with functional drift.

Proposition 14 *Assume that there is a sequence of progressively measurable functionals $b_n : [0, T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d, n \geq 1$ fulfilling (2), (3). Further suppose that the following conditions hold:*
(i) $b_n, n \geq 1$ are smooth, tempered, non-anticipating functions (see Definition 4).
(ii) The sequence $b_n, n \geq 1$ satisfies the integrability (34) and

$$\sup_{n \geq 1} E\left[\exp\left(4\int_0^T \left\| b_n^{\shortmid}(t, B.) \right\|_{L(\mathcal{W}^d, \mathbb{R}^d)} dt\right)\right] < \infty \tag{57}$$

is valid.
(iii) The factor R_n in (35) converges to zero for a progressively measurable $b : [0, T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$.
Then
$$Y_t^{(b_n)} \longrightarrow Y_t^{(b)} \text{ in } L^2(\mu) \text{ uniformly in } t \text{ as } n \longrightarrow \infty.$$
Furthermore, $Y_t^{(b)}$ is Malliavin differentiable for all $t \geq 0$.

Proof. It is shown in [KS] that (i) and (ii) of Proposition (14) entail that the solutions $X_t^{(n)}$ of (26) with respect to $b_n(t, \cdot), n \geq 1$ are Malliavin differentiable and that in virtue of Gronwall's Lemma

$$E\left[\int_0^T \left\| D_s X_t^{(n)} \right\|^2 ds\right] \leq const.E\left[\exp\left(2\int_0^T \left\| b_n^{\shortmid}(t, X_\cdot^{(n)}) \right\|_{L(\mathcal{W}^d, \mathbb{R}^d)} dt\right)\right]$$

for $0 \leq t \leq T$. Further Girsanov's theorem in connection with (34) yields

$$E\left[\int_0^T \left\| D_s X_t^{(n)} \right\|^2 ds\right] \leq const.E_{\widehat{\mu}}^{\frac{1}{2}}\left[\exp\left(4\int_0^T \left\| b_n^{\shortmid}(t, \widehat{B}.) \right\|_{L(\mathcal{W}^d, \mathbb{R}^d)} dt\right)\right]$$

for all $0 \leq t \leq T$, where $\widehat{\mu}$, $\widehat{B}$ are copies of μ, B, respectively.

Then the proof follows from Theorem 6, Lemma 10 and a weak compactness argument in Hilbert spaces. ■

We need to define the following classes $\mathcal{L}$,$\mathcal{M}$ and $\mathcal{R}$ of approximating functionals:

Definition 15 *(i)The class $\mathcal{L}$ consists of all progressively measurable $b : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ for which there exists a sequence of approximating functionals $b_n : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ in the sense of (37) such that the conditions of Theorem 6 are fulfilled. Recall that for such b_n*

$$\underline{\lim}_{n,m\longrightarrow\infty} L(n,m) \left\| Y_t^{i,b_n} - Y_t^{i,b_m} \right\|_{\mathcal{N}^{\frac{1}{2}}}^{\frac{3}{2}} \longrightarrow 0 \tag{58}$$

with the "weight"

$$L(n,m) := \left\| Y_t^{i,b_n} - Y_t^{i,b_m} \right\|_{\mathcal{N}^{-\frac{3}{2}}}^{\frac{1}{2}} \underset{n,m\longrightarrow\infty}{\longrightarrow} 0$$

holds.

(ii) We shall denote by $\mathcal{M}$ the class of all progressively measurable $b : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ such that there exists a sequence of functionals $b_n : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ which satisfies the assumptions of Proposition 14. Thus we observe that $\mathcal{M}$ is a subclass of $\mathcal{L}$.

(iii) Further $\mathcal{R}$ is defined as the collection of all progressively measurable $\varphi : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ such that the factor R_n in (35) converges to zero for a sequence of continuous functionals $\varphi_n : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$.

Before we come to our main results we send ahead the following auxiliary result:

Lemma 16 *Assume that $b : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ is contained in the class $\mathcal{L}$. Then the transformation property (28) also extends to the process Y_t^b, that is*

$$\varphi^{(i)}\left(t, Y_\cdot^b\right) = E_{\widehat{\mu}}\left[\varphi^{(i)}\left(t, \widehat{B}_\cdot\right) \mathcal{E}_T^{\diamond}(b)\right] \tag{59}$$

a.e. for all $0 \leq t \leq T, i = 1, ..., d$ and $\varphi = (\varphi^{(1)}, ..., \varphi^{(d)}) \in \mathcal{R}$.

Proof. Lemma 9 shows that $Y_t^b \in L^2(\mu;\mathbb{R}^d)$ for $0 \leq t \leq T$. Let us first verify that the process $Y_t^b, 0 \leq t \leq T$ has a continuous modification. By assumption we know that there exists a sequence of progressively measurable $b_n : [0,T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d, n \geq 1$ such that the integrability condition (34) holds and such that the factor (35) in Theorem 6 tends to zero. Since each $Y_t^{b_n}$ is a strong solution of the SDE (26) with respect to the drift b_n we obtain from Girsanov's theorem and (34) that

$$\begin{aligned} E_\mu\left[\left(Y_t^{i,b_n} - Y_u^{i,b_n}\right)^2\right] &= E_{\widehat{\mu}}\left[\left(\widehat{B}_t^{(i)} - \widehat{B}_s^{(i)}\right)^2 \mathcal{E}\left(\int_0^T b_n(s, \widehat{B}_\cdot) d\widehat{B}_s\right)\right] \\ &\leq const. \left|t - u\right| \end{aligned}$$

for all $0 \leq u, t \leq T$, $n \geq 1, i = 1, ..., d$. By Theorem 6 we have that

$$Y_t^{(b_n)} \longrightarrow Y_t^{(b)} \text{ in } L^2(\mu) \tag{60}$$

for all $0 \leq t \leq T$. The latter implies that

$$E_{\mu}\left[\left(Y_t^{i,b} - Y_u^{i,b}\right)^2\right] \leq const. \left|t-u\right| \tag{61}$$

for all $0 \leq u, t \leq T, i = 1, ..., d$. Then Kolmogorov's Lemma provides a continuous modification of Y_t^b.

Consider the operators $B_m = (B_m^{(1)}, ..., B_m^{(d)})$ on $\mathcal{W}^d$, where the components are given by the Bernstein polynomials $B_m^{(i)}$:

$$(B_m^{(i)}x)(t) = \sum_{k=0}^{m} x^{(i)}\left(\frac{Tk}{m}\right)\binom{m}{k}t^k(T-t)^{m-k}, 0 \leq t \leq T, i = 1, ..., d.$$

Then it is well-known that

$$B_m x \longrightarrow x \text{ for } m \longrightarrow \infty \text{ in } \mathcal{W}^d \tag{62}$$

for all $x \in \mathcal{W}^d$. Let $\varphi_l, l \geq 1$ be an sequence which approximates φ in the sense of the definition of $\mathcal{R}$. Using a truncation argument we can assume without loss of generality that φ and $\varphi_l, l \geq 1$ are uniformly bounded. Then Proposition 1 gives the representation

$$\varphi_l\left(t, B_m Y^{b_n}\right) = E_{\widehat{\mu}}\left[\varphi_l\left(t, B_m \widehat{B}_{\cdot}\right)\mathcal{E}_T^{\diamond}(b_n)\right] \tag{63}$$

for all l, n, m. Using dominated convergence we see from (60) that $\varphi_l\left(t, B_m Y^{b_n}\right)$ tends to $\varphi_l\left(t, B_m Y^b\right)$ for $n \longrightarrow \infty$ in $(\mathcal{S})^*$. But the right hand side of (63) converges to $E_{\widehat{\mu}}\left[\varphi_l\left(t, B_m \widehat{B}_{\cdot}\right)\mathcal{E}_T^{\diamond}(b)\right]$ in $(\mathcal{S})^*$ in virtue of an analogous estimate to (41). Thus

$$\varphi_l\left(t, B_m Y^b\right) = E_{\widehat{\mu}}\left[\varphi_l\left(t, B_m \widehat{B}_{\cdot}\right)\mathcal{E}_T^{\diamond}(b)\right]$$

for all l, m. On the other hand dominated convergence, the $\mathcal{S}$-transform and (62) yield for $m \longrightarrow \infty$

$$\varphi_l\left(t, Y_{\cdot}^b\right) = E_{\widehat{\mu}}\left[\varphi_l\left(t, \widehat{B}_{\cdot}\right)\mathcal{E}_T^{\diamond}(b)\right]$$

for all l. Finally, applying dominated convergence and the $\mathcal{S}$-transform once more we get for $l \longrightarrow \infty$ the desired result. ■

We can now state our first main result.

Theorem 17 *Suppose that $b : [0, T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ belongs to the class $\mathcal{L}$ of Definition 15. Then there exists a strong solution X_t of*

$$dX_t = b(t, X_{\cdot})dt + dB_t, \;\; X_0 = x.$$

The solution X_t can be approximated in $L^2(\mu)$, uniformly in t by solutions of (26) with respect to regular drifts in the sense of the definition of $\mathcal{L}$. Moreover, X_t takes the explicit form (33). If in addition the solutions of (26) are unique in law, then strong uniqueness holds.

Proof. We shall employ the transformation property (59) of Lemma 16 to verify that Y_t^b is a unique strong solution of the SDE (26): For convenience we set $x = 0$. Since $\widehat{B}_t$ is a weak solution of (26) for the drift $b(s, \cdot) + \xi(s)$ with respect to the measure $d\mu^* = \mathcal{E}\left(\int_0^T \left(b(s, \widehat{B}.) + \xi(s)\right) d\widehat{B}_s\right) d\widehat{\mu}$ we obtain that

$$
\begin{aligned}
&S(Y_t^{i,b})(\xi) = E_{\widehat{\mu}}\left[\widehat{B}_t^{(i)}\mathcal{E}\left(\int_0^T \left(b(s, \widehat{B}.) + \xi(s)\right) d\widehat{B}_s\right)\right] \\
&= E_{\mu^*}\left[\widehat{B}_t^{(i)}\right] \\
&= E_{\mu^*}\left[\int_0^t \left(b^{(i)}(s, \widehat{B}.) + \xi^{(i)}(s)\right) ds\right] \\
&= \int_0^t E_{\widehat{\mu}}\left[b^{(i)}(s, \widehat{B}.)\mathcal{E}\left(\int_0^T \left(b(u, \widehat{B}.) + \xi(u)\right) d\widehat{B}_u\right)\right] ds + S\left(B_t^{(i)}\right)(\xi).
\end{aligned}
$$

Thus by the transformation property (59) applied to b it follows that

$$
S(Y_t^{i,b})(\xi) = S(\int_0^t b(u, Y_\cdot^{i,b})du)(\xi) + S(B_t^{(i)})(\xi).
$$

So the characterization theorem (i.e. injectivity of S) gives that

$$
Y_t^b = \int_0^t b(s, Y_\cdot^b)ds + B_t.
$$

If the solutions are unique in law, we will be able to apply Girsanov's theorem under condition (34) for $n = 0$ and hence all strong solutions will necessarily take the form (28). Thus Y_t^b is the unique strong solution of (26).

Note that Y_t^b as an approximation of adapted solutions in $L^2(\mu; \mathbb{R}^d)$ is also adapted, since our underlying filtration is $\mu-$augmented (see (27)). This completes the proof. ■

The next result shows that drifts in (26) restricted to the class $\mathcal{M} \subseteq \mathcal{L}$ of Definition 15 even yield regular strongs solutions.

Theorem 18 *Require that $b : [0, T] \times \mathcal{W}^d \longrightarrow \mathbb{R}^d$ is contained in the class $\mathcal{M} \subseteq \mathcal{L}$ of Definition 15. Then there exists a Malliavin differentiable strong solution X_t of*

$$
dX_t = b(t, X.)dt + dB_t, \ X_0 = x.
$$

Moreover the solution is explicitly given by (33).

Proof. Since $\mathcal{M} \subseteq \mathcal{L}$ Theorem 17 entails the existence of a strong solution X_t. The Malliavin differentiability of X_t follows from the fact that X_t can be approximated by a sequence of solutions of the SDE (26) which is bounded with respect to the Sobolev norm $\|\cdot\|_{2,1}$ in (4). ■

Now let us focus on solutions of the SDE (26) with measurable drifts $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$. We shall give deterministic integrability conditions on b to ensure Malliavin differentiable strong solutions of (26).

Theorem 19 *Assume that $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ can be approximated by a sequence of functions $b_n : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, n \geq 1$ from the class $\mathcal{O}$ under the conditions (37) in Theorem 6 and (46), (47) in Proposition 12 (or (56) in Corollary 13). Then there exists a Malliavin differentiable strong solution X_t of*

$$dX_t = b(t, X_t)dt + dB_t, \; X_0 = x.$$

Moreover, X_t has the explicit representation (33).

Proof. As it can be seen from the proof of Proposition 12 the conditions (46) and (47) (or (56)) imply that the approximating sequence of solutions $X_t^{(n)}$ is bounded with respect to the norm $\|\cdot\|_{2,1}$. By Lemma 10 we conclude that b belongs to the class $\mathcal{L}$ Therefore there exists a Malliavin differentiable strong solution of (26). ■

Remark 20 *In Theorem 17, 18 and 19 we obtain uniqueness in law by demanding e.g. that all weak solutions Y_t of (26) satisfy*

$$\int_0^T |b(t, Y_\cdot)|^2 \, dt < \infty \;\; a.e.$$

See e.g. [KS, Proposition 5.3.10].

Remark 21 *Zvonkin [Zv] and Veretennikov [V] were able to prove the existence of a unique strong solution of (26), if b is bounded and measurable. See e.g. also [GK], [KR] and [GM]. However the latter authors' techniques do not apply to the functional case as treated in Theorem 17 or 18. Furthermore Theorem 19 (and 18) yields solutions of (26) for a "rich" class of measurable drift coefficients, which turn out to be Malliavin differentiable. In the one-dimensional case the authors in [M-BP] prove under (34) the existence of a unique strong solution $X_t \in L^2(\mu)$ of (26). Just as in this paper the authors employ the white noise representation in Theorem 1 to construct solutions. However their approximation argument heavily relies on a comparison result for one-dimensional SDE's driven by a Brownian motion and does not give existence of solutions in a smaller subspace of $L^2(\mu)$.*

Remark 22 *One can reduce the factor 512 in the integrability condition (34) to $\frac{1}{2} + \varepsilon$ for arbitrarily small $\varepsilon > 0$ by applying Hölder's inequality in the previous proofs more carefully.*

Corollary 23 *Retain the conditions of Theorem 19. Further assume that the drift coefficient*

b is time-homogeneous and that $b \in L^2_{loc}(\mathbb{R})$. Then X_t of Theorem 19 is a unique strong solution in $\mathbb{D}_{1,2}$.

Proof. Let Y_t be a solution of (26). Denote by $L_t^Y(x)$ the local time of the continuous semimartingale Y_t, $0 \leq t \leq T$. Then the occupation density formula supplies

$$\int_0^T |b(Y_s)|^2 \, ds = \int_{\mathbb{R}} |b(x)|^2 \, L_T^Y(x) dx.$$

Since $(x \longmapsto L_t^Y(x))$ is RCCL with compact support $\mu-$a.e. (see e.g. [Be]), it follows from $b \in L^2_{loc}(\mathbb{R})$ that

$$\int_0^T |b(x)|^2 \, ds < \infty \;\; \widehat{\mu}\text{-a.e.}$$

Then we infer from [KS, Proposition 5.3.10] that X_t is unique in law. However the latter gives strong uniqueness. ■

Let us finally extend Theorem 19 to a class of non-degenerate $d-$dimensional Itô-diffusions.

Theorem 24 *Consider the time-homogeneous $\mathbb{R}^d-$valued SDE*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \;\; X_0 = x \in \mathbb{R}^d, \;\; 0 \leq t \leq T, \tag{64}$$

where the coefficients $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable. Suppose that there exists a bijection $\Lambda : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, which is twice continuously differentiable. Denoting by $\Lambda_x : \mathbb{R}^d \longrightarrow L\left(\mathbb{R}^d, \mathbb{R}^d\right)$ and $\Lambda_{xx} : \mathbb{R}^d \longrightarrow L\left(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d\right)$ the corresponding derivatives of Λ require that

$$\Lambda_x(y)\sigma(y) = id_{\mathbb{R}^d} \text{ for } y \text{ a.e.}$$

as well as

$$\Lambda^{-1} \text{ is Lipschitz continuous.}$$

Further we impose on the function

$$\begin{aligned} & b_*(x) \\ : \;\; = \; & \Lambda_x\left(\Lambda^{-1}(x)\right)\left[b(\Lambda^{-1}(x))\right] + \frac{1}{2}\Lambda_{xx}\left(\Lambda^{-1}(x)\right)\left[\sum_{i=1}^d \sigma(\Lambda^{-1}(x))[e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(x))[e_i]\right] \end{aligned}$$

to fulfill the conditions of Theorem 19, where e_i, $i = 1, ..., d$ is a basis of $\mathbb{R}^d$. Then there exists a Malliavin differentiable solution X_t of (64).

Proof. By assumption we can apply Itô's Lemma to (64) and obtain

$$\begin{aligned} & dY_t \\ = \; & \Lambda_x\left(\Lambda^{-1}(Y_t)\right)\left[b(\Lambda^{-1}(Y_t))\right] + \frac{1}{2}\Lambda_{xx}\left(\Lambda^{-1}(Y_t)\right)\left[\sum_{i=1}^d \sigma(\Lambda^{-1}(Y_t))[e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(Y_t))[e_i]\right] dt \\ & + dB_t, \\ Y_0 \; = \; & \Lambda(x), \;\; 0 \leq t \leq T, \end{aligned}$$

where $Y_t = \Lambda(X_t)$. From Theorem 19 we know that there exists a Malliavin differentiable solution Y_t of the above equation. Hence $X_t = \Lambda^{-1}(Y_t)$ solves (64). Finally, since Λ^{-1} is Lipschitz continuous X_t is Malliavin differentiable. See e.g. [N, Proposition 1.2.3]. ■

Remark 25 *Let us mention that the results of this Section do not depend on the special choice of the probability space* $(\Omega, \mathcal{F}, \mu)$. *Actually as long as we have a complete probability space* $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$, *where* $\widetilde{\mathcal{F}}$ *is generated by a* $\widetilde{P}-$*Brownian motion* $\widetilde{B}$, *we can build up a white noise theory based on chaos expansions of iterated integrals. More specifically, if* X_t *is a strong solution of (64) with respect to* $(\Omega, \mathcal{F}, \mu)$ *we are able to lift the strong solution to the space* $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$ *by using the "lifting operator"* $\Phi : L^2(\mu) \longrightarrow L^2(P)$ *defined by*

$$\Phi(\xi)(\widetilde{\omega}) = E_\mu \left[\xi(\omega) \exp^{\widetilde{\diamond}} \left(\sum_{j=1}^{d} \int_0^T \widetilde{W}_s^{(j)}(\widetilde{\omega}) dB_s^{(j)}(\omega) - \frac{1}{2} \int_0^T \left(\widetilde{W}_s^{(j)} \right)^{\widetilde{\diamond} 2} (\widetilde{\omega}) dB_s^{(j)}(\omega) \right) \right], \tag{65}$$

where $\widetilde{W}_s$ *is the white noise and* $\widetilde{\diamond}$ *the Wick product with respect to the space* $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$. *So* Φ *applied to* X_t, *that is* $Y_t = \Phi(X_t)$ *provides a strong solution on* $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$. *The latter can be easily seen by employing the* S*-transform. Note that the operator (65) exhibits an explicit way to transform a functional* $f(B.)$ *into* $f(\widetilde{B}.)$ *by replacing the corresponding Brownian motion.*

4 Solutions of SDE's in spaces of smooth random variables

In this section we shall show that strong solutions of the SDE (26) with respect to a certain class of drift coefficients actually live in the spaces $\mathcal{C}_q$. Recall from Section 2.2 that each $f \in \mathcal{C}_q$ has a chaos expansion which can be exponentially weighted (see 24). Therefore $\mathcal{C}_q$ is contained in the space $\mathbb{D}_{\infty,2} = \underset{k \longrightarrow \infty}{proj\ lim} \mathbb{D}_{k,2}$ for all $q > 0$. Note that the space $\mathcal{C}$ introduced in Section 2.2 is a projective limit of the Hilbert spaces $\mathcal{C}_q$.

Theorem 26 *Let* X_t *be a strong solution of the SDE (26). Assume for the measurable drift* $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *in (26) that all spatial partial derivatives of* b *exist and that*

$$\left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_1^{\alpha_n}} b(t,x) \right| \leq K \tag{66}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$, $\alpha_1, ..., \alpha_n \in \mathbb{N}_0$ *with* $|\alpha| = \alpha_1 + ... + \alpha_n$, $n \geq 1$ *and a constant* $K > 0$. *Then for all* $q > 0$ *there exists a* $T > 0$ *such that*

$$X_t \in \mathcal{C}_q$$

for all $0 \leq t \leq T$.

Proof. Without loss of generality we shall give the proof for $d = 1$. Note that our assumptions on the drift b imply that X_t belongs to the Meyer-Watanabe test function space $\mathbb{D}_\infty$. See [W]. Let us first assume that $T \cdot K \cdot e^K < 1$ for $K \geq 2$. We subdivide the proof into the following two steps:

1. We want to show that

$$E\left[\|D_\cdot^n X_t\|^2_{L^2([0,T]^n)} \right] \leq T^n (n!)^2 K^{6n-2} e^{2Kn} \tag{67}$$

for all $0 \leq t \leq T$, where D^n denotes the $n-$th iterated Malliavin derivative. For this purpose we want to prove by induction that

$$\left|D^n_{s_1,\dots,s_n} X_t\right| \leq \frac{1}{2}\varphi(n) K^n e^{Kn} \tag{68}$$

as well as

$$\left|D^n_{s_1,\dots,s_n} b(t,X_t)\right| \leq \varphi(n) K^{n+1} e^{Kn} \tag{69}$$

for all $0 \leq t \leq T$, $0 \leq s_1,\dots,s_n,t \leq T$, $n \geq 1$, where $\varphi(n)$ satisfies the recursion

$$\varphi(n) = \sum_{j=0}^{n-2} \binom{n-1}{j} \varphi(j+1)\varphi(n-1-j), \varphi(1) := 1. \tag{70}$$

We shall prove the estimate (68) by induction. Since $D_s X_t$ solves the linear equation (50) we obtain that

$$D_s X_t = \exp\left(\int_s^t \frac{\partial}{\partial x} b(u,X_u)du\right),\ 0 \leq s \leq t.$$

So

$$|D_s X_t| \leq e^{TK} \leq \frac{1}{2} K e^K.$$

On the other hand we have

$$|D_s b(t,X_t)| = \left|\frac{\partial}{\partial x} b(t,X_t) D_s X_t\right| \leq K e^{TK} \leq K^2 e^K.$$

Since $\varphi(1) = 1$ the estimates (68) and (69) hold for $n = 1$. Now suppose that (68) and (69) are valid for $n > 1$. Let us observe that $D^n_{\cdot} b(t,X_t)$ takes the form

$$D^n_{\cdot} b(t,X_t) = \sum_{j=0}^{n-1} \binom{n-1}{j} \left(D^{j+1}_{\cdot} X_t\right)\left(D^{n-1-j}_{\cdot} b(t,X_t)\right), \tag{71}$$

where we use the convention $D^0 b(t,X_t) = \frac{\partial}{\partial x} b(t,X_t)$. By applying the Malliavin derivative repeatedly to both sides of the linear equation (50) we see that

$$D^{n+1}_{\cdot} X_t = \int_{\cdot}^t \sum_{j=0}^{n-1} \binom{n}{j} \left(D^{j+1}_{\cdot} X_u\right)\left(D^{n-j}_{\cdot} b(t,X_u)\right) du + \int_{\cdot}^t \frac{\partial}{\partial x} b(t,X_u) D^{n+1}_{\cdot} X_u du.$$

Then using the induction hypothesis and (70) gives

$$\begin{aligned}
\left|D^{n+1}_{\cdot} X_t\right| &\leq T \cdot \sum_{j=0}^{n-1} \binom{n}{j} \sup_{0\leq u\leq T} \left|D^{j+1}_{\cdot} X_u\right| \cdot \sup_{0\leq u\leq T} \left|D^{n-j}_{\cdot} b(t,X_t)\right| \\
&\quad + \int_0^t K \cdot \left|D^{n+1}_{\cdot} X_u\right| du \\
&\leq T \cdot \frac{1}{2} \cdot \sum_{j=0}^{n-1} \binom{n}{j} \varphi(j+1) K^{j+1} e^{K(j+1)} \cdot \varphi(n-j) K^{n-j+1} e^{K(n-j)} \\
&\quad + \int_0^t K \cdot \left|D^{n+1}_{\cdot} X_u\right| du \\
&= \frac{1}{2} T K^{n+2} e^{K(n+1)} \varphi(n+1) + \int_0^t K \cdot \left|D^{n+1}_{\cdot} X_u\right| du
\end{aligned} \tag{72}$$

By Gronwall's Lemma it follows that

$$\begin{aligned}\left|D_{\cdot}^{n+1}X_t\right| &\leq \frac{1}{2}TK^{n+2}e^{K(n+1)}\varphi(n+1)e^K \\ &\leq \frac{1}{2}K^{n+1}e^{K(n+1)}\varphi(n+1).\end{aligned}$$

On the other hand, by invoking the relation (71), the last estimate, the recursion (70) and the induction hypothesis we get

$$\begin{aligned}\left|D_{\cdot}^{n+1}b(t,X_t)\right| &\leq \sum_{j=0}^{n-1}\binom{n}{j}\left|D_{\cdot}^{j+1}X_u\right|\cdot\left|D_{\cdot}^{n-j}b(t,X_t)\right| + \left|\frac{\partial}{\partial x}b(t,X_t)\right|\left|D_{\cdot}^{n+1}X_u\right| \\ &\leq \sum_{j=0}^{n-1}\binom{n}{j}\frac{1}{2}\varphi(j+1)K^{j+1}e^{K(j+1)}\varphi(n-j)K^{n-j+1}e^{K(n-j)} \\ &\quad +K\cdot\frac{1}{2}K^{n+1}e^{K(n+1)}\varphi(n+1) \\ &= \frac{1}{2}K^{n+2}e^{K(n+1)}\varphi(n+1) + K\cdot\frac{1}{2}K^{n+1}e^{K(n+1)}\varphi(n+1) \\ &= K^{n+2}e^{K(n+1)}\varphi(n+1),\end{aligned}$$

which completes the induction for (68) and (69). In order to show the estimate (67) set $\widetilde{\varphi}(n) = n\varphi(n)$. Then

$$\widetilde{\varphi}(n) = \sum_{j=1}^{n-1}\binom{n}{j}\widetilde{\varphi}(j)\frac{\widetilde{\varphi}(n-j)}{(n-j)}, \widetilde{\varphi}(1) = 1.$$

One observes that $\widetilde{\varphi}(n) = n!\Lambda(n)$ is the solution with

$$\Lambda(n) = \sum_{j=1}^{n-1}\frac{1}{n-j}\Lambda(j)\Lambda(n-j), \Lambda(1) = 1.$$

So $\Lambda(n) \leq C(n)$, where $C(n)$ are the Catalan numbers given by

$$C(n) = \sum_{j=1}^{n-1}C(j)C(n-j), C(1) = 1.$$

Since $C(n) = \frac{1}{2n-1}\binom{2n-1}{n-1} \leq 2^{2n-1}$ it follows that

$$\varphi(n) \leq n!2^{2n-1}.$$

Thus (67) is an immediate consequence from the latter inequality and (68), (69).

2. We wish to prove that for all $q > 0$ there exists a time horizon $T = T(q) < 1$ such that

$$\|X_t\|_{\mathcal{C}_q}^2 = \left\|e^{q\sqrt{N}}X_t\right\|_{L^2(\mu)}^2 < \infty \tag{73}$$

for all $0 \leq t \leq T$, where $\|\cdot\|_{\mathcal{C}q}$ are the norms on $\mathcal{C}_q$ and N the number operator. See (24).

Set $C = -\sqrt{N}$. Then Meyer's inequality [N, Theorem 1.5.1] gives

$$E\left[|C^n X_t|^2\right] \leq B(n)\left(E\left[\|D_\cdot^n X_t\|^2_{L^2([0,T]^n)}\right] + E\left[X_t^2\right]\right) \tag{74}$$

for a constant $B(n)$ depending on n, $0 \leq t \leq T$. In checking the proof of Meyer's inequality carefully [N, Theorem 1.5.1] it turns out that the constant $B(n)$ in (74) can be chosen as

$$B(n) = M^{n-1} \cdot \prod_{j=1}^{n-1}\left(1 + \frac{1}{j}\right)^{\frac{j}{2}},\ n \geq 1 \tag{75}$$

for a universal constant M. So we observe that

$$B(n) \leq M^{n-1} e^{\frac{n-1}{2}},\ n \geq 1.$$

Using (74) in connection with (75), the spectral theorem for selfadjoint operators and the inequality (67) permits the estimate

$$\begin{aligned}
\|X_t\|^2_{\mathcal{C}_q} &= \left\|e^{q\sqrt{N}} X_t\right\|^2_{L^2(\mu)} \leq \sum_{n\geq 0} \frac{(2q)^n}{n!} E^{\frac{1}{2}}\left[|C^n X_t|^2\right] E^{\frac{1}{2}}\left[X_t^2\right] \\
&\leq E^{\frac{1}{2}}\left[X_t^2\right] \sum_{n\geq 0} \frac{(2q)^n}{n!} M^{\frac{n-1}{2}} e^{\frac{n-1}{4}} \left(E^{\frac{1}{2}}\left[\|D_\cdot^n X_t\|^2_{L^2([0,T]^n)}\right] + E^{\frac{1}{2}}\left[X_t^2\right]\right) \\
&\leq Q^2 e^{2q} + Q \sum_{n\geq 0} \frac{(2q)^n}{n!} M^{\frac{n-1}{2}} e^{\frac{n-1}{4}} T^{\frac{n}{2}} n! K^{3n-2} e^{Kn} \\
&\leq Q^2 e^{2q} + Q \sum_{n\geq 0} \left(2q\sqrt{M}\sqrt[4]{e} K^3 e^K \sqrt{T}\right)^n < \infty,
\end{aligned}$$

for all $0 \leq t \leq T$ and a constant Q provided $T < \frac{1}{\left(2q\sqrt{M}\sqrt[4]{e}K^3 e^K\right)^2}$, which completes the proof. ■

Theorem 26 and Lemma 10 indicate that one can construct a "larger" class of solutions of (26) in the spaces $\mathcal{C}_q$.

Theorem 27 *Assume that the sequence of measurable functions $b_n : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d, n \geq 0$ with $b_0 = b$ satisfies the conditions (34) and (37) in Theorem 6. Further suppose that the solutions $X_t^{(n)}$ of (26) with respect to the drift b_n, $n \geq 1$ are contained in $\mathcal{C}_{2q+p}$ for some $q, p > 0$. Set $L(n,m) = \left\|X_t^{(n)} - X_t^{(m)}\right\|_{\mathcal{C}_{-p}}$. Then*

$$L(n,m) \underset{n,m\longrightarrow\infty}{\longrightarrow} 0$$

for all $t \geq 0$. Let us require that

$$L(n,m) \cdot \left\|X_t^{(n)} - X_t^{(m)}\right\|_{\mathcal{C}_{2q+p}} \underset{n,m\longrightarrow\infty}{\longrightarrow} 0 \tag{76}$$

for all $0 \leq t \leq T$. Then there exists a strong solution X_t of (26) such that

$$X_t \in \mathcal{C}_q$$

for all $0 \leq t \leq T$. Moreover X_t is explicitly given by the process Y_t^b defined in (33).

Proof. Theorem 1 implies that the solutions $X_t^{(n)}$ coincide with the processes $Y_t^{b_n}, n \geq 1$ defined in (33). Then one observes that

$$\begin{aligned}
\left\| X_t^{(n)} - X_t^{(m)} \right\|_{\mathcal{C}_q}^2 &= \left\| Y_t^{b_n} - Y_t^{b_m} \right\|_{\mathcal{C}_q}^2 \\
&\leq \left\| Y_t^{b_n} - Y_t^{b_m} \right\|_{\mathcal{C}_{-p}} \left\| Y_t^{b_n} - Y_t^{b_m} \right\|_{\mathcal{C}_{2q+p}} \\
&= L(n,m) \cdot \left\| Y_t^{b_n} - Y_t^{b_m} \right\|_{\mathcal{C}_{2q+p}} \\
&\underset{n,m \longrightarrow \infty}{\longrightarrow} 0
\end{aligned}$$

for all $q \geq 0$. Lemma 10 shows that

$$L(n,m) \longrightarrow 0$$

for $m, n \longrightarrow \infty$ for $p > 0$. So we get that

$$\left\| Y_t^{b_n} - Y_t^{b_m} \right\|_{\mathcal{C}_q} \longrightarrow 0$$

for $m, n \longrightarrow \infty$ for $q \geq 0$. Since $\mathcal{C}_q$ is a Hilbert space bounded sets of $\mathcal{C}_q$ are weakly relative compact. Then by checking the proof of Lemma 16 we see that

$$Y_t^{b_n} \underset{n \longrightarrow \infty}{\longrightarrow} Y_t^b \text{ in } \mathcal{C}_q$$

and that Y_t^b satisfies the transformation property (59). Using the latter property (59) just as in the proof of Theorem 17 gives the result. ■

Let us now consider the space $\mathcal{C}_{q,\infty} \subseteq \mathcal{C}_q$ with Fréchet topology induced by the norms

$$\|f\|_{\mathcal{C}_q^p} := \left\| e^{q\sqrt{N}} f \right\|_{L^p(\mu;\mathbb{R}^d)}, \ p > 0 \tag{77}$$

for fixed $q > 0$.

From the multiplier theorem [IK, Lemma 8.2] it follows that $\mathcal{C}_{q,\infty}$ is contained in the Meyer-Watanabe test function space $\mathbb{D}_\infty$. The next result is a refinement of Theorem 26.

Proposition 28 *Let X_t be a strong solution of the SDE (26) with a drift $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ as in Theorem 26. Then for all $q > 0$ there exists a time horizon T such that*

$$X_t \in \mathcal{C}_{q,\infty}$$

for all $0 \leq t \leq T$.

Proof. Let $d = 1$. Using the estimate 68 we obtain that

$$E^{\frac{1}{p}} \left[\|D_\cdot^n X_t\|_{L^2([0,T]^n)}^p \right] \leq T^{\frac{n}{2}} n! K^{3n-1} e^{Kn} \tag{78}$$

for all $n \geq 0, 0 \leq t \leq T$. Then Meyer's inequality (see e.g. [N, Theorem 1.5.1]) together with (78) gives

$$\|X_t\|_{\mathcal{C}_q^p} = \left\| e^{q\sqrt{N}} X_t \right\|_{L^p(\mu)} \leq \sum_{n \geq 0} \frac{q^n}{n!} M_p^{n-1} e^{\frac{n-1}{2}} T^{\frac{n}{2}} n! K^{3n-1} e^{Kn} < \infty$$

for $0 \leq t \leq T$ with T sufficiently small. Here we applied a bound similar to (75). The general case T can be covered by proceeding as in the proof of Theorem 26. ■

5 Discussion

Our approach as presented in the previous sections exhibits potential to cover a variety of other types of stochastic equations. For example this technique can be used to inquire into the following problems:

1 Infinite dimensional Brownian motion with drift:

$$dX_t = b(t, X_t)dt + dB_t^Q, \ X_0 = x \in H, \tag{79}$$

where B_t^Q is a Q-cylindrical Brownian motion on a Hilbert space H and Q a positive symmetric trace class operator. This case requires a modification of the proof of Lemma 10, since parts of its proof are based on arguments in $\mathbb{R}^d$. We demonstrate in [P1] how to cope with this problem. It is also conceivable to include a densely defined operator in the drift term of (79).

2 Infinite dimensional jump SDE's:

$$dX_t = \gamma(X_{t-})dL_t, \ L_0 = x \in H, \tag{80}$$

where L_t is a $H-$valued additive process. Using a white noise framework for additve processes it is possible to construct strong solutions of (80) under integrability conditions on γ in terms of the compensator of the jump measure of L_t. See [P2], where the case of an H-valued stable Lévy process L_t is studied.

3 Certain types of anticipative SDE's, that is e.g. Brownian motion with non-adapted drift.

4 Similar equations for fractional Brownian motion or fractional Lévy processes.

Finally let us summarize some advantages of our method:

(i) As mentioned in the Introduction we propose a constructive method to determine strong solutions of stochastic equations. Employing an approximation technique in the spaces $\mathbf{D}_{1,2}$ or $\mathcal{C}_q$ we directly show that an explicitly defined generalized process solves the stochastic equation. Thus we obtain strong existence (and uniqueness) of solutions without resorting to the celebrated theorem of Yamada-Watanabe. This result states that the existence of a weak solution (on some probability space with some Wiener process) and the pathwise uniqueness entail existence of a strong solution, that is

$$\text{weak existence } + \text{ pathwise uniqueness} \Longrightarrow \text{ strong uniqueness.}$$

Many authors in literature first construct a weak solution by using e.g. stopping time methods or the Skorohod embedding technique. Then they use pathwise uniqueness to retrieve a unique strong solution. See [GK], [GM], [KR] and the references therein. Our method is diametrically opposed to Yamada-Watanabe in the sense that we prove:

$$\text{strong existence } + \text{ uniqueness in law} \Longrightarrow \text{ strong uniqueness}$$

(ii) Our method can be used to study the functional SDE

$$dX_t = b(t, X.)dt + dB_t. \tag{81}$$

In general this case is more delicate than the Euclidean one, since one has to find conditions to avoid collisions with the example of Tsirel'son. Tsirel'son [Ts] gave an example of a uniformly bounded progressively measurable path functional b, which rules out the existence of a strong solution of (81). This was also an important counterexample in connection with innovation problems in filtering theory. As far as we can see the techniques of the authors [GK], [GM], [KR] are not applicable to the "infinite dimensional" equation (81). As we pointed out our method also works for other types of driving processes.

(iii) We also show that solutions of a larger class of Itô diffusions are Malliavin differentiable. This feature is attractive and yields dividends in various applications. See e.g. [N] for an account of important applications.

(iv) It is conceivable to extend our method to more general driving processes than additive processes by using e.g. a white noise theory based on biorthogonal chaos decompositions. See e.g. [KSS].

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